

IPPP/04/66
DCPT/04/132
FERMILAB-PUB-04-281-T
hep-ph/0411071

Effective Hamiltonian for Non-Leptonic $|\Delta F| = 1$ Decays at NNLO in QCD

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Abstract

We compute the effective hamiltonian for non-leptonic $|\Delta F| = 1$ decays in the standard model including next-to-next-to-leading order QCD corrections. In particular, we present the complete three-loop anomalous dimension matrix describing the mixing of current-current and QCD penguin operators. The calculation is performed in an operator basis which allows to consistently use fully anticommuting γ_5 in dimensional regularization at an arbitrary number of loops. The renormalization scheme dependences and their cancellation in physical quantities is discussed in detail. Furthermore, we demonstrate how our results are transformed to a different basis of effective operators which is frequently adopted in phenomenological applications. We give all necessary two-loop constant terms which allow to obtain the three-loop anomalous dimensions and the corresponding initial conditions of the two-loop Wilson coefficients in the latter scheme. Finally, we solve the renormalization group equation and give the analytic expressions for the low-energy Wilson coefficients relevant for non-leptonic B meson decays beyond next-to-leading order in both renormalization schemes.

1 Introduction

Perturbative QCD effects have an important impact on the structure of the effective hamiltonian for non-leptonic $|\Delta F| = 1$ processes with $F = S, C$ or B , which describes the weak decay of the corresponding mesons and hadrons. Most notably, they can lead to a sizable enhancement of the $\Delta I = 1/2$ transitions, of the CP -violating ratio ϵ'/ϵ , and of the QCD penguin contributions to rare and radiative B decays within the Standard Model (SM) [1] and some of its innumerable extensions [2–7].

In all cases, these short-distance QCD effects can be systematically calculated using an effective field theory framework, which allows to resum large QCD logarithms of the form $L \equiv \ln \mu/M_w$ by solving the Renormalization Group Equation (RGE) that governs the scale dependence of the Wilson coefficient functions of the relevant $|\Delta F| = 1$ local operators built out of the light and massless SM fields. After the pioneering Leading Order (LO) calculation of the $O(\alpha_s^n L^n)$ contributions [8], the resummation of the $O(\alpha_s^n L^{n-1})$ logarithms has been completed more than ten years ago and subsequently confirmed by several groups. The main components of the perturbative Next-to-Leading Order (NLO) calculation are *i)* the one-loop $O(\alpha_s)$ corrections to the relevant Wilson coefficient functions [9–11] and *ii)* the two-loop $O(\alpha_s^2)$ Anomalous Dimension Matrix (ADM) describing the mixing of the associated physical operators [10–15].

To improve on the present NLO calculation, one needs to include one more order in the strong coupling expansion, aiming at a resummation of all the $O(\alpha_s^n L^{n-2})$ logarithmic enhanced corrections. The completion of this Next-to-Next-to-Leading Order (NNLO) computation constitutes the core of this work. Since the two-loop $O(\alpha_s^2)$ matching corrections to the relevant Wilson coefficients are already known from [16] the only missing ingredient to perform this task is the knowledge of the three-loop $O(\alpha_s^3)$ ADM describing the mixing of the current-current and QCD penguin operators. In this paper we will close this gap by employing standard techniques [15, 17, 18] to carry out a direct calculation of the required ADM adopting the renormalization scheme introduced in [11]. Since in this scheme the QCD penguin operators are defined in such a way that traces with γ_5 do not occur to all orders in perturbation theory, we are allowed to consistently use dimensional regularization with a naive anticommuting γ_5 . This feature is very welcome, as it makes the actual three-loop calculation completely automatic and rather straightforward.

The NNLO ADM we have computed can be used in analyses of new physics models as well, provided they do not introduce new operators with respect to the SM. This applies, for example, to the case of the two Higgs doublet models [2], to some supersymmetric scenarios with minimal flavor violation [3], and to specific models of universal extra dimensions [4]. On the other hand, in left-right-symmetric models [5] and in the general supersymmetric SM [6], additional operators with different chirality structures arise [19]. In many cases one can exploit the chiral invariance of QCD and use the same ADM, but in general an extended basis is required.

Another strong motivation to write the article at hand was that the three-loop ADM computed here is part and parcel of the complete NNLO analysis of rare semi-leptonic $\bar{B} \rightarrow X_s \ell^+ \ell^-$ decays presented recently by us in collaboration with Christoph Bobeth and Paolo Gambino [20]. Furthermore, it constitutes a integral part of the NNLO calculation of radiative $\bar{B} \rightarrow X_s \gamma$ decays, admittedly a very ambitious enterprise, which nevertheless has already aroused the interest of some theorists [21]. Whereas our previous work dealt exclusively with the phenomenological application of our result, we will now — in the spirit of [9–11] — focus on the more formal aspects of the renormalization of effective field theories, such as the issue of scheme dependences in general, their cancellation in physical observables, and the general transformation properties of the ADM and the Wilson coefficients under a change of scheme. In this respect we will extend the existing NLO results [9, 10] to the next order, paying special attention to the conceptual features related to the renormalization of the strong coupling constant.

While much of the discussion in this paper is therefore rather technical, our general results have important practical applications. This will be showcased by means of a couple of examples. In particular, we will devote a sizable part of the present article to derive the explicit NNLO relation between our and a different renormalization scheme that is commonly used in the literature on weak decays [1]. In the latter scheme, which we shall call “traditional” scheme from now on, the applied form of the effective hamiltonian introduces unwanted traces containing γ_5 by definition. These traces turn out to be harmless at the LO, but involve a lot of technical difficulties related to the use of Fierz symmetry arguments in $n = 4 - 2\epsilon$ dimensions at the NLO [10, 14]. Applying the same scheme in a direct calculation of the ADM at the NNLO or even beyond would thus be extremely tedious. We will not attempt such a direct computation here, but avail the derived NNLO relation between our and the “traditional” scheme to find the NNLO ADM and the corresponding matching conditions in the latter scheme on detours. As another exercise we solve the RGE and give the analytic expressions for the $\Delta B = -\Delta S = 1$ low-energy Wilson coefficients beyond NLO in both renormalization schemes. We are aware of the fact that some of the formulas presented below are rather long. Nevertheless we believe that at least some of them should be useful to the reader interested mainly in the application of the presented formalism to weak decays rather than in the conceptual subtleties, which obviously address more technical minded colleagues.

The main part of this paper is organized as follows: in Section 2 we present the general structure of the effective hamiltonian for non-leptonic $|\Delta F| = 1$ decays at the NNLO level. Section 3 is devoted to the simplest application of the general formalism, namely the $\Delta B = -\Delta S = 1$ decays. We recall the relevant effective hamiltonian and list all the dimension-five and six operators that will be needed in the calculation of the three-loop ADM. In Section 4 we collect the results for the initial conditions of the relevant Wilson coefficients through NNLO obtained in [11, 16]. After a brief description of the actual three-loop calculation the final result for the ADM is given in Section 5. In Section 6 we solve the RGE to find the explicit NNLO expressions for the low-energy Wilson coefficients

relevant for non-leptonic B meson decays. In Section 7 we elaborate on the question of scheme dependence related to the renormalization of the effective operators as well as the strong coupling constant. Section 8 starts out with a general discussion of the non-trivial nature of a change of the basis of physical operators in the framework of dimensional regularization, followed by a demonstration of how the NNLO results for the ADM and the matching conditions are transformed to the “traditional” basis of effective operators. Finally, in Section 9 we summarize the main results of this work.

Some technical details as well as additional material has been relegated to the appendices: in Appendix A.1 we derive the explicit form of the matrix kernels that are needed to find the evolution matrices through NNLO, while Appendices A.2 and A.3 contain all ingredients that are necessary to transform our results to the “traditional” set of operators. The NNLO analytic formulas for the low-energy Wilson coefficients relevant for non-leptonic B meson decays in the latter scheme will be given in Appendix A.4, which concludes our paper.

2 General Structure

The effective hamiltonian for non-leptonic $|\Delta F| = 1$ decays has the following generic structure [1]

$$\mathcal{H}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{\text{CKM}} \vec{Q}^T \vec{C}(\mu). \quad (1)$$

Here G_F denotes the Fermi constant and \vec{Q}^T is a row vector containing the relevant local operators Q_i . Explicit expressions will be given in Section 3. $\vec{C}(\mu)$ is a column vector containing the Wilson coefficients $C_i(\mu)$ that together with the Cabibbo-Kobayashi-Maskawa (CKM) factor [22] V_{CKM} describe the strength with which a given operator enters the hamiltonian, and μ is the renormalization scale. The decay amplitude for a decay of a meson M into a final state F is simply given by $\langle F | \mathcal{H}_{\text{eff}} | M \rangle$.

The Wilson coefficient functions evolve from the initial scale μ_0 down to μ , which in practical applications is much lower than μ_0 , according to their RGE. Using dimensional regularization with $n = 4 - 2\epsilon$ and considering only mass independent renormalization schemes it is given by

$$\mu \frac{d}{d\mu} \vec{C}(\mu) = \hat{\gamma}^T(g) \vec{C}(\mu), \quad (2)$$

where $\hat{\gamma}(g)$ is the ADM corresponding to \vec{Q} . Neglecting all electromagnetic effects, the general solution of this equation reads

$$\vec{C}(\mu) = \hat{U}(\mu, \mu_0) \vec{C}(\mu_0), \quad (3)$$

with

$$\hat{U}(\mu, \mu_0) = T_g \exp \int_{g(\mu_0)}^{g(\mu)} dg' \frac{\hat{\gamma}^T(g')}{\beta(g')}, \quad (4)$$

$$\hat{\gamma}(g) = \sum_{i=0}^{\infty} \left(\frac{g^2}{16\pi^2} \right)^{i+1} \hat{\gamma}^{(i)}, \quad \text{and} \quad \beta(g) = -g \sum_{i=0}^{\infty} \left(\frac{g^2}{16\pi^2} \right)^{i+1} \beta_i. \quad (5)$$

Here $\vec{C}(\mu_0)$ are the initial conditions of the evolution and T_g denotes ordering of the coupling constants $g(\mu)$ in such a way that their value increases from right to left. $\beta(g)$ is the QCD β function.

Keeping the first three terms in the expansions of $\hat{\gamma}(g)$ and $\beta(g)$ as given in Eq. (5), we find for the evolution matrix $\hat{U}(\mu, \mu_0)$ in the NNLO approximation

$$\hat{U}(\mu, \mu_0) = \hat{K}(\mu) \hat{U}^{(0)}(\mu, \mu_0) \hat{K}^{-1}(\mu_0), \quad (6)$$

where

$$\begin{aligned} \hat{K}(\mu) &= \hat{1} + \frac{\alpha_s(\mu)}{4\pi} \hat{J}^{(1)} + \left(\frac{\alpha_s(\mu)}{4\pi} \right)^2 \hat{J}^{(2)}, \\ \hat{K}^{-1}(\mu_0) &= \hat{1} - \frac{\alpha_s(\mu_0)}{4\pi} \hat{J}^{(1)} - \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^2 \left(\hat{J}^{(2)} - (\hat{J}^{(1)})^2 \right), \end{aligned} \quad (7)$$

and

$$\hat{U}^{(0)}(\mu, \mu_0) = \hat{V} \text{diag} \left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \right)^{a_i} \hat{V}^{-1}, \quad (8)$$

denotes the LO evolution matrix, which depends on the matrix \hat{V} and the so-called magic numbers a_i that are obtained via diagonalizing $\hat{\gamma}^{(0)T}$:

$$\left(\hat{V}^{-1} \hat{\gamma}^{(0)T} \hat{V} \right)_{ij} = 2\beta_0 a_i \delta_{ij}. \quad (9)$$

In order to give the explicit expressions for the matrices $\hat{J}^{(1)}$ and $\hat{J}^{(2)}$ we define

$$\hat{J}^{(i)} = \hat{V} \hat{S}^{(i)} \hat{V}^{-1}, \quad \text{and} \quad \hat{G}^{(i)} = \hat{V}^{-1} \hat{\gamma}^{(i)T} \hat{V}, \quad (10)$$

for $i = 1, 2$. The entries of the matrix kernels $\hat{S}^{(1)}$ and $\hat{S}^{(2)}$ are given by

$$\begin{aligned} S_{ij}^{(1)} &= \frac{\beta_1}{\beta_0} a_i \delta_{ij} - \frac{G_{ij}^{(1)}}{2\beta_0 (1 + a_i - a_j)}, \\ S_{ij}^{(2)} &= \frac{\beta_2}{2\beta_0} a_i \delta_{ij} + \sum_k \frac{1 + a_i - a_k}{2 + a_i - a_j} \left(S_{ik}^{(1)} S_{kj}^{(1)} - \frac{\beta_1}{\beta_0} S_{ij}^{(1)} \delta_{jk} \right) - \frac{G_{ij}^{(2)}}{2\beta_0 (2 + a_i - a_j)}, \end{aligned} \quad (11)$$

where the first line recalls the classical NLO result [9], and the second one represents the corresponding NNLO expression, in agreement with [23]. The explicit derivation of $\hat{S}^{(1)}$ and $\hat{S}^{(2)}$ is presented in Appendix A.1.

Let us now recall how the initial conditions of the Wilson coefficients are obtained. The amplitude for a given non-leptonic quark decay is calculated perturbatively in the full theory including all possible diagrams such as W -boson exchange, QCD penguin and box diagrams as well as gluon corrections to all these building blocks. The result up to the NNLO is given schematically by

$$\mathcal{A}_{\text{full}} = \langle \vec{Q} \rangle^{(0)T} \left(\vec{A}^{(0)} + \frac{\alpha_s(\mu_0)}{4\pi} \vec{A}^{(1)} + \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^2 \vec{A}^{(2)} \right), \quad (12)$$

where $\langle \vec{Q} \rangle^{(0)}$ denotes the tree-level matrix elements of \vec{Q} .

A second step involves the calculation of the decay amplitude in the QCD effective theory. It generally requires the computation of the operator insertions into current-current and QCD penguin diagrams of the effective theory together with gluon corrections to these insertions. Including QCD corrections up to the NNLO one finds

$$\mathcal{A}_{\text{eff}} = \langle \vec{Q} \rangle^{(0)T} \left(\hat{1} + \frac{\alpha_s(\mu_0)}{4\pi} \hat{r}^{(1)T} + \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^2 \hat{r}^{(2)T} \right) \vec{C}(\mu_0), \quad (13)$$

where the matrices $\hat{r}^{(1)}$ and $\hat{r}^{(2)}$ codify the one- and two-loop matrix elements of \vec{Q} , respectively.

The matching procedure between full and effective theory establishes the initial conditions $\vec{C}(\mu_0)$ for the Wilson coefficients. Equating $\mathcal{A}_{\text{full}}$ and \mathcal{A}_{eff} in Eqs. (12) and (13) at a scale μ_0 translates into the following identity [24]

$$\begin{aligned} \vec{C}(\mu_0) &= \vec{A}^{(0)} + \frac{\alpha_s(\mu_0)}{4\pi} \left(\vec{A}^{(1)} - \hat{r}^{(1)T} \vec{A}^{(0)} \right) \\ &+ \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^2 \left(\vec{A}^{(2)} - \hat{r}^{(1)T} \left[\vec{A}^{(1)} - \hat{r}^{(1)T} \vec{A}^{(0)} \right] - \hat{r}^{(2)T} \vec{A}^{(0)} \right). \end{aligned} \quad (14)$$

Combining Eqs. (2), (6), (7) and (14), we finally obtain

$$\begin{aligned} \vec{C}(\mu) &= \hat{K}(\mu) \hat{U}^{(0)}(\mu, \mu_0) \left(\vec{A}^{(0)} + \frac{\alpha_s(\mu_0)}{4\pi} \left[\vec{A}^{(1)} - \hat{R}^{(1)} \vec{A}^{(0)} \right] \right. \\ &\left. + \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^2 \left[\vec{A}^{(2)} - \hat{R}^{(1)} \vec{A}^{(1)} - \left(\hat{R}^{(2)} - (\hat{R}^{(1)})^2 \right) \vec{A}^{(0)} \right] \right), \end{aligned} \quad (15)$$

where

$$\hat{R}^{(1)} = \hat{r}^{(1)T} + \hat{J}^{(1)}, \quad \text{and} \quad \hat{R}^{(2)} = \hat{r}^{(2)T} + \hat{J}^{(2)} + \hat{r}^{(1)T} \hat{J}^{(1)}, \quad (16)$$

are certain combinations of $\hat{r}^{(1)T}$, $\hat{r}^{(2)T}$, $\hat{J}^{(1)}$ and $\hat{J}^{(2)}$ which will play a special role in Section 7 where we will discuss the issue of renormalization scheme dependences in detail.

3 Effective Hamiltonian for $\Delta B = -\Delta S = 1$ Decays

The simplest application of the general formalism outlined in the previous section is the case of non-leptonic B meson decays governed by the $b \rightarrow s$ transition. For definiteness we will therefore give explicit formulas for the $\Delta B = -\Delta S = 1$ decays only. However, it is straightforward to transform them to the other $|\Delta F| = 1$ cases. Neglecting contributions proportional to the small CKM factor $V_{us}^* V_{ub}$ which are irrelevant here, the corresponding effective off-shell hamiltonian is given by

$$\mathcal{H}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left(\vec{Q}^T \vec{C}(\mu) + \vec{N}^T \vec{C}_N(\mu) + \vec{B}^T \vec{C}_B(\mu) + \vec{E}^T \vec{C}_E(\mu) \right). \quad (17)$$

In addition to the gauge-invariant operators \vec{Q} , non-physical operators arise as counterterms in the renormalization of higher loop One-Particle-Irreducible (1PI) off-shell Green's functions with insertions of the operators \vec{Q} . These non-physical operators can in general be divided into three different classes [13, 25–27]: *i*) operators \vec{N} that vanish by use of the QCD Equations Of Motion (EOM), *ii*) non-physical counterterms \vec{B} that can be written as a Becchi-Rouet-Stora-Tyutin (BRST) variation [28] of some other operators — so-called BRST-exact operators — and *iii*) evanescent operators \vec{E} that vanish algebraically in $n = 4$ dimensions.

The set of physical operators \vec{Q} consists of six dimension-six operators, which can be chosen as [11, 29]

$$\begin{aligned} Q_1 &= (\bar{s}_L \gamma_{\mu_1} T^a c_L) (\bar{c}_L \gamma^{\mu_1} T^a b_L), \\ Q_2 &= (\bar{s}_L \gamma_{\mu_1} c_L) (\bar{c}_L \gamma^{\mu_1} b_L), \\ Q_3 &= (\bar{s}_L \gamma_{\mu_1} b_L) \sum_q (\bar{q} \gamma^{\mu_1} q), \\ Q_4 &= (\bar{s}_L \gamma_{\mu_1} T^a b_L) \sum_q (\bar{q} \gamma^{\mu_1} T^a q), \\ Q_5 &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3} b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3} q), \\ Q_6 &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3} T^a b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3} T^a q), \end{aligned} \quad (18)$$

and one dimension-five operator

$$Q_8 = \frac{1}{g} m_b (\bar{s}_L \sigma^{\mu_1 \mu_2} T^a b_R) G_{\mu_1 \mu_2}^a. \quad (19)$$

Here we have used the definitions $\gamma_{\mu_1 \dots \mu_m} \equiv \gamma_{\mu_1} \dots \gamma_{\mu_m}$, $\gamma^{\mu_1 \dots \mu_m} \equiv \gamma^{\mu_1} \dots \gamma^{\mu_m}$ and $\sigma^{\mu_1 \mu_2} \equiv i [\gamma^{\mu_1}, \gamma^{\mu_2}]/2$, and the sum over q extends over all light quark flavors. g is the strong coupling constant, q_L and q_R are the chiral quark fields, $G_{\mu_1 \mu_2}^a$ is the gluonic field strength tensor, and T^a are the generators of $SU(3)_C$, normalized so that $\text{Tr}(T^a T^b) \equiv \delta^{ab}/2$.

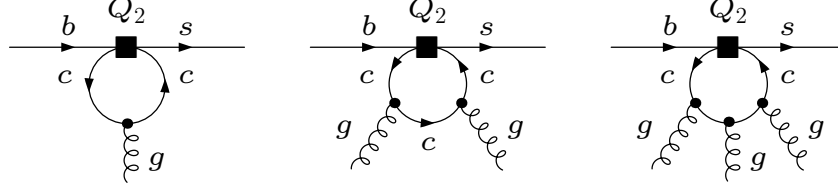


Figure 1: The one-loop 1PI diagrams which mix Q_2 into $N_1^{(1)}$.

The physical operators given in Eqs. (18) and (19) include the current-current operators Q_1 and Q_2 , the QCD penguin operators Q_3 – Q_6 and the chromomagnetic moment operator Q_8 . Notice that we have defined Q_1 – Q_6 in such a way that problems connected with the treatment of γ_5 in $n = 4 - 2\epsilon$ dimensions do not arise [11]. Consequently, we are allowed to consistently use a fully anticommuting γ_5 in dimensional regularization throughout the calculation.

As far as the EOM-vanishing operators are concerned, the specific structure of only one of them [11]

$$N_1^{(1)} = \frac{1}{g} \bar{s}_L \gamma^{\mu_1} T^a b_L D^{\mu_2} G_{\mu_1 \mu_2}^a + Q_4, \quad (20)$$

is relevant in finding the one- and two-loop mixing of the four-quark operators Q_1 – Q_6 . The corresponding divergent one-loop 1PI diagrams are shown in Figure 1.

In order to remove the ultraviolet (UV) divergences related to the two-loop subdiagrams with insertions of Q_1 – Q_6 depicted in Figure 2, another ten EOM-vanishing operators need to be considered [15]

$$\begin{aligned} N_1^{(2)} &= \frac{1}{g^2} m_b \bar{s}_L \not{D} \not{D} b_R, \\ N_2^{(2)} &= \frac{i}{g^2} \bar{s}_L \not{D} \not{D} \not{D} b_L, \\ N_3^{(2)} &= \frac{i}{g} \left[\bar{s}_L \overleftarrow{\not{D}} \sigma^{\mu_1 \mu_2} T^a b_L G_{\mu_1 \mu_2}^a - G_{\mu_1 \mu_2}^a \bar{s}_L T^a \sigma^{\mu_1 \mu_2} \not{D} b_L \right] + Q_8, \\ N_4^{(2)} &= \frac{i}{g} m_b \bar{s}_L \left[\overleftarrow{\not{D}} \not{G} - \not{G} \not{D} \right] b_R, \\ N_5^{(2)} &= i \left[\bar{s}_L \left(\overleftarrow{\not{D}} \not{G} \not{G} - \not{G} \not{G} \not{D} \right) b_L - i m_b \bar{s}_L \not{G} \not{G} b_R \right], \\ N_6^{(2)} &= \frac{1}{g} \left[\bar{s}_L \left(\overleftarrow{\not{D}} \overleftarrow{\not{D}} \not{G} + \not{G} \not{D} \not{D} \right) b_L + i m_b \bar{s}_L \not{G} \not{D} b_R \right], \\ N_7^{(2)} &= i \left[\bar{s}_L \left(\overleftarrow{\not{D}} G_{\mu_1}^a G^{a \mu_1} - G_{\mu_1}^a G^{a \mu_1} \not{D} \right) b_L - i m_b \bar{s}_L G_{\mu_1}^a G^{a \mu_1} b_R \right], \end{aligned}$$

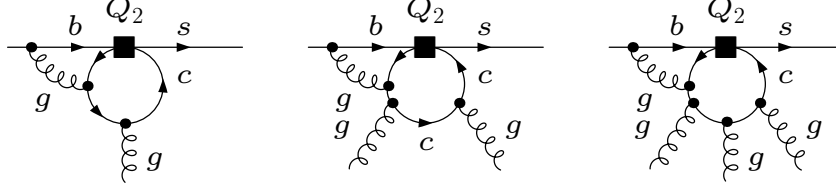


Figure 2: Some of the two-loop 1PI diagrams which mix Q_2 into $N_1^{(1)}$ and $N_1^{(2)} - N_{10}^{(2)}$.

$$\begin{aligned}
N_8^{(2)} &= \frac{1}{g} \left[\bar{s}_L \left(\overleftarrow{D} \overleftarrow{D}_{\mu_1} G^{\mu_1} + G_{\mu_1} D^{\mu_1} \overleftarrow{D} \right) b_L + im_b \bar{s}_L G_{\mu_1} D^{\mu_1} b_R \right], \\
N_9^{(2)} &= \frac{1}{g} \left[\bar{s}_L \overleftarrow{D} \not{G} \not{D} b_L + im_b \bar{s}_L \overleftarrow{D} \not{G} b_R \right], \\
N_{10}^{(2)} &= d^{abc} \left[\bar{s}_L \left(\overleftarrow{D} T^a - T^a \overleftarrow{D} \right) b_L - im_b \bar{s}_L T^a b_R \right] G_{\mu_1}^b G^{c\mu_1}, \tag{21}
\end{aligned}$$

where $D_\mu \equiv \partial_\mu + igG_\mu$ and $\overleftarrow{D}_\mu \equiv \overleftarrow{\partial}_\mu - igG_\mu$ denotes the covariant derivative of the gauge group $SU(3)_C$ acting on the fields to the right and left, respectively. G_μ^a denotes the gluon field, and we have used the definitions $G_\mu \equiv G_\mu^a T^a$ and $d^{abc} \equiv 2\text{Tr}(\{T^a, T^b\}T^c)$.

It is important to remark that the EOM-vanishing operators introduced in Eqs. (20) and (21) arise as counterterms independently of what kind of infrared (IR) regularization is adopted in the computation of the anomalous dimensions of Q_1 – Q_6 . However, if the regularization respects the underlying symmetry, and all the diagrams are calculated on-shell, non-physical operators have vanishing matrix elements [25–27, 30]. In this case the EOM-vanishing operators given in Eqs. (20) and (21) play no role in the calculation of the mixing of physical operators. If the gauge symmetry is broken this is no longer the case, as diagrams with insertions of non-physical operators will generally have non-vanishing projection on the physical operators. As we will discuss in Section 5, our IR regularization implies a massive gluon propagator, and therefore non-physical counterterms play a crucial role at intermediate stages of the anomalous dimensions calculation.

In contrast to the case of the two-loop mixing of the magnetic operators considered in [15, 17], it is a priori not clear if BRST-exact operators do arise as counterterms of Q_1 – Q_6 . Since the BRST variation raises both ghost number and mass dimension by one unit, it is evident that any BRST-exact operator that potentially could mix with Q_1 – Q_6 has to be a BRST variation of a dimension-five operator containing a single anti-ghost field. The only possibility for the latter operator having the correct chirality structure is given in the R_ξ gauge by [26]

$$B_1^{(2)} = s \left[\frac{1}{g} (\partial_{\mu_1} \bar{u}^a) (\bar{s}_L \gamma^{\mu_1} T^a b_L) \right] = -\frac{1}{g} \left[\frac{1}{\xi} \partial_{\mu_1} \partial^{\mu_2} G_{\mu_2}^a - g f^{abc} (\partial_{\mu_1} \bar{u}^b) u^c \right] (\bar{s}_L \gamma^{\mu_1} T^a b_L), \tag{22}$$

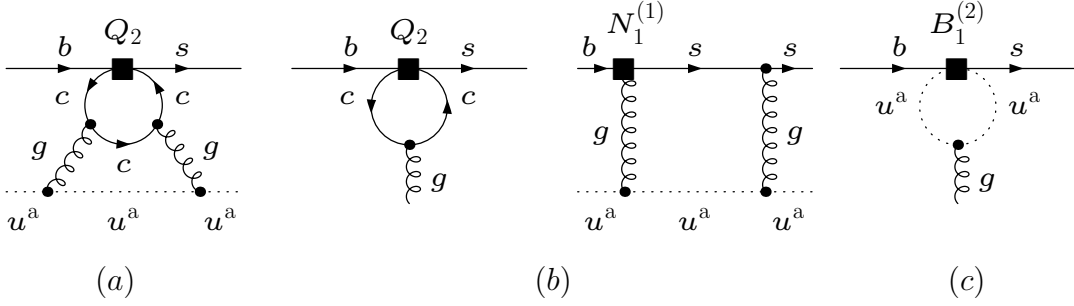


Figure 3: (a) A typical example of a divergent two-loop 1PI diagram which potentially could introduce a mixing of Q_2 into $B_1^{(2)}$. (b) A typical example of a counterterm contribution needed to renormalize the corresponding two-loop 1PI diagrams. (c) The contribution to the one-loop matrix element of $B_1^{(2)}$ containing an effective $b \rightarrow su^a \bar{u}^a$ vertex, which has a non-vanishing on-shell projection on Q_4 if a non-zero ghost mass is used in the calculation. The QCD penguin and box contributions to the matrix element that contain an effective $b \rightarrow sg$ vertex are not shown.

where s denotes the BRST operator, u^a and \bar{u}^a are the ghost and anti-ghost fields, f^{abc} are the totally antisymmetric structure constants of $SU(3)_C$ and ξ is the covariant gauge-parameter.

Although there is no obvious reason why $B_1^{(2)}$ should not appear as a counterterm of Q_1 – Q_6 , it turns out that up to three loops $B_1^{(2)}$ does not play a role in the mixing of physical operators considered in the paper at hand. The key observation thereby is that the overall contribution from the two-loop 1PI diagrams depicted in Figure 3 (a) is canceled by the corresponding counterterm contribution as shown in Figure 3 (b), so that the associated renormalization constant is exactly zero at $O(\alpha_s^2)$. Therefore $B_1^{(2)}$ does not contribute to the mixing of Q_1 – Q_6 into Q_4 , although its one-loop $O(\alpha_s)$ matrix element displayed in Figure 3 (c) does not vanish if it is computed using non-vanishing gluon and ghost masses to regulate IR divergences.

In order to remove the divergences of all possible 1PI Green's functions with single insertion of Q_1 – Q_6 we have to introduce some evanescent operators \vec{E} as well. At the one-loop level one encounters four evanescent operators, which can be chosen to be [11, 29]

$$\begin{aligned}
 E_1^{(1)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3} T^a c_L) (\bar{c}_L \gamma^{\mu_1 \mu_2 \mu_3} T^a b_L) - 16Q_1, \\
 E_2^{(1)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3} c_L) (\bar{c}_L \gamma^{\mu_1 \mu_2 \mu_3} b_L) - 16Q_2, \\
 E_3^{(1)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} q) + 64Q_3 - 20Q_5, \\
 E_4^{(1)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} T^a b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} T^a q) + 64Q_4 - 20Q_6.
 \end{aligned} \tag{23}$$

At the two-loop level four more evanescent operators do arise, that can be defined as [11, 29]

$$E_1^{(2)} = (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} T^a c_L) (\bar{c}_L \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} T^a b_L) - 256Q_1 - 20E_1^{(1)},$$

$$\begin{aligned}
E_2^{(2)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} c_L) (\bar{c}_L \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} b_L) - 256 Q_2 - 20 E_2^{(1)}, \\
E_3^{(2)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7} b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7} q) + 1280 Q_3 - 336 Q_5, \\
E_4^{(2)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7} T^a b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7} T^a q) + 1280 Q_4 - 336 Q_6. \quad (24)
\end{aligned}$$

Finally, at the three-loop level another four evanescent operators are needed. We define them in the following way:

$$\begin{aligned}
E_1^{(3)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7} T^a c_L) (\bar{c}_L \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7} T^a b_L) - 4096 Q_1 - 336 E_1^{(1)}, \\
E_2^{(3)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7} c_L) (\bar{c}_L \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7} b_L) - 4096 Q_2 - 336 E_2^{(1)}, \\
E_3^{(3)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7 \mu_8 \mu_9} b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7 \mu_8 \mu_9} q) + 21504 Q_3 - 5440 Q_5, \\
E_4^{(3)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7 \mu_8 \mu_9} T^a b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7 \mu_8 \mu_9} T^a q) + 21504 Q_4 - 5440 Q_6. \quad (25)
\end{aligned}$$

Needless to say, the above choice of evanescent operators $E_1^{(3)}-E_4^{(3)}$ is not unique, in the sense that their particular structure can be changed quite a lot without affecting the three-loop ADM of the four-quark operators Q_1-Q_6 . For instance, adding any multiple of ϵ times any physical operator to them leaves the ADM unchanged up to $O(\alpha_s^3)$. This is in contrast to what happens if such a redefinition is applied to the one- and two-loop evanescent operators given in Eqs. (23) and (24).

4 Initial Conditions of the Wilson Coefficients

Let us now turn to the initial conditions $\vec{C}(\mu_0)$ of the Wilson coefficients. Their values are found by matching the full to the effective theory amplitudes perturbatively in α_s . The NLO and NNLO approximation requires the calculation of one- and two-loop diagrams both in the SM and the low-energy effective theory. Some of the SM two-loop 1PI diagrams one has to consider in order to find the $O(\alpha_s^2)$ corrections to $\vec{C}(\mu_0)$ are displayed in Figure 4. Restricting ourselves to the physical on-shell operators Q_1-Q_6 and setting $\mu_0 = M_W$, one obtains using dimensional regularization with a naive anticommuting γ_5 [11, 16]:

$$\begin{aligned}
C_1(M_W) &= 15 \frac{\alpha_s(M_W)}{4\pi} + \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 \left(\frac{7987}{72} + \frac{17}{3} \pi^2 - \tilde{T}_0(x_t) \right), \\
C_2(M_W) &= 1 + \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 \left(\frac{127}{18} + \frac{4}{3} \pi^2 \right), \\
C_3(M_W) &= \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 \tilde{G}_1(x_t), \\
C_4(M_W) &= \frac{\alpha_s(M_W)}{4\pi} \tilde{E}_0(x_t) + \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 \tilde{E}_1(x_t),
\end{aligned}$$

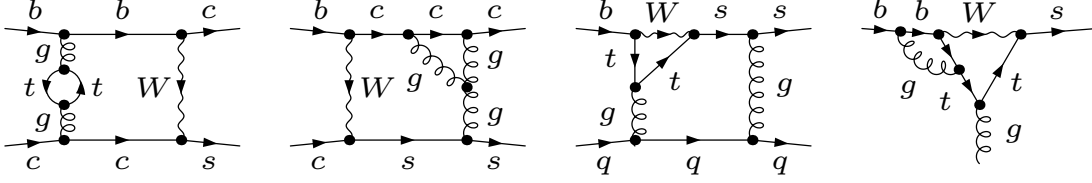


Figure 4: Some of the SM two-loop 1PI diagrams one has to calculate in order to find the Wilson coefficients of the four-quark operators Q_1 – Q_6 at $O(\alpha_s^2)$.

$$\begin{aligned}
C_5(M_W) &= \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 \left(\frac{14}{135} + \frac{2}{15} \tilde{E}_0(x_t) - \frac{1}{10} \tilde{G}_1(x_t) \right), \\
C_6(M_W) &= \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 \left(\frac{7}{36} + \frac{1}{4} \tilde{E}_0(x_t) - \frac{3}{16} \tilde{G}_1(x_t) \right),
\end{aligned} \tag{26}$$

where $x_t = m_t^2/M_W^2$. The one-loop Inami-Lim [31] function $\tilde{E}_0(x_t)$ characterizing the effective off-shell vertex involving a gluon reads

$$\tilde{E}_0(x_t) = \frac{8 - 42x_t + 35x_t^2 - 7x_t^3}{12(x_t - 1)^3} - \frac{4 - 16x_t + 9x_t^2}{6(x_t - 1)^4} \ln x_t. \tag{27}$$

The one-loop function $\tilde{T}_0(x_t)$ originates from diagrams like the first one shown in Figure 4. Subtracting the corresponding terms in the gluon propagator in the so-called momentum space subtraction scheme at $q^2 = 0$, which guarantees that α_s has the same numerical value on the full and effective side at the matching scale through NNLO, one finds [16]

$$\begin{aligned}
\tilde{T}_0(x_t) &= \frac{112}{9} + 32x_t + \left(\frac{20}{3} + 16x_t \right) \ln x_t \\
&\quad - (8 + 16x_t) \sqrt{4x_t - 1} \text{Cl}_2 \left(2 \arcsin \left(\frac{1}{2\sqrt{x_t}} \right) \right),
\end{aligned} \tag{28}$$

with $\text{Cl}_2(x) = \text{Im}[\text{Li}_2(e^{ix})]$ and $\text{Li}_2(x) = -\int_0^x dt \ln(1-t)/t$. The remaining two-loop functions $\tilde{E}_1(x_t)$ and $\tilde{G}_1(x_t)$ take the following form [16]

$$\begin{aligned}
\tilde{E}_1(x_t) &= -\frac{1120 - 12044x_t - 5121x_t^2 - 5068x_t^3 + 7289x_t^4}{648(x_t - 1)^4} \\
&\quad - \frac{380 - 7324x_t + 17702x_t^2 + 2002x_t^3 - 5981x_t^4 + 133x_t^5}{324(x_t - 1)^5} \ln x_t \\
&\quad - \frac{112 - 530x_t - 3479x_t^2 + 2783x_t^3 - 1129x_t^4 + 515x_t^5}{108(x_t - 1)^5} \ln^2 x_t \\
&\quad - \frac{40 - 190x_t - 81x_t^2 - 614x_t^3 + 515x_t^4}{54(x_t - 1)^4} \text{Li}_2(1 - x_t) + \frac{10}{81} \pi^2,
\end{aligned}$$

$$\begin{aligned}
\tilde{G}_1(x_t) = & \frac{554 - 2523x_t + 2919x_t^2 - 662x_t^3 + 243(x_t - 1)^3}{27(x_t - 1)^4} \\
& + \frac{88 - 142x_t - 357x_t^2 + 100x_t^3 + 35x_t^4}{81(x_t - 1)^4} \ln x_t - \frac{20 - 40x_t + 5x_t^2}{27(x_t - 1)^2} \ln^2 x_t \\
& + \frac{40 - 160x_t - 30x_t^2 + 100x_t^3 - 10x_t^4}{27(x_t - 1)^4} \text{Li}_2(1 - x_t) - \frac{20}{81} \pi^2.
\end{aligned} \tag{29}$$

5 Anomalous Dimension Matrix

Before presenting our results for the anomalous dimensions describing the mixing of the four-quark operators Q_1 – Q_6 up to $O(\alpha_s^3)$ let us recall some definitions that will turn out to be useful in the rest of the paper.

Upon renormalization the bare Wilson coefficients $\vec{C}_B(\mu)$ of Eq. (1) transform as

$$\vec{C}_B(\mu) = \hat{Z}^T \vec{C}(\mu). \tag{30}$$

In terms of the renormalization constant matrix \hat{Z} the ADM of Eq. (2) is then given by

$$\hat{\gamma}(g) = \hat{Z} \mu \frac{d}{d\mu} \hat{Z}^{-1}. \tag{31}$$

The renormalization constants Z_{ij} of the operator Q_j can be expanded in powers of g in the following way

$$Z_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} \left(\frac{g^2}{16\pi^2} \right)^k Z_{ij}^{(k)}, \quad \text{with} \quad Z_{ij}^{(k)} = \sum_{l=0}^k \frac{1}{\epsilon^l} Z_{ij}^{(k,l)}. \tag{32}$$

Following the standard $\overline{\text{MS}}$ scheme prescription, Z_{ij} is given by pure $1/\epsilon^l$ poles, except when i corresponds to an evanescent operator and j does not. In the latter case, the renormalization constant is finite, to make sure that the matrix elements of the evanescent operators vanish in $n = 4$ dimensions [13, 27].

In a mass independent renormalization scheme the only μ -dependence of Z_{ij} resides in the coupling constant. In consequence, we might rewrite Eq. (31) as

$$\gamma_{ij} = \beta(\epsilon, g) Z_{ik} \frac{d}{dg} Z_{kj}^{-1}, \tag{33}$$

where $\beta(\epsilon, g)$ is related to the usual QCD β function via

$$\beta(\epsilon, g) = -\epsilon g + \beta(g). \tag{34}$$

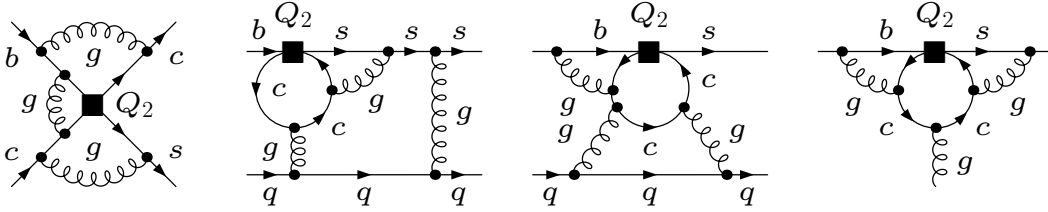


Figure 5: Some of the three-loop 1PI diagrams we had to calculate in order to find the mixing among the four-quark operators Q_1 – Q_6 at $O(\alpha_s^3)$.

The finite parts of Eq. (33) in the limit of ϵ going to zero give the anomalous dimensions. Inserting the expansions of $\hat{\gamma}(g)$ and $\beta(g)$ in powers of g , as given in Eq. (5), one immediately finds [15, 18] for the anomalous dimensions governing the evolution of physical operators up to third order in the strong coupling parameter:

$$\begin{aligned}\hat{\gamma}^{(0)} &= 2\hat{Z}^{(1,1)}, \\ \hat{\gamma}^{(1)} &= 4\hat{Z}^{(2,1)} - 2\hat{Z}^{(1,1)}\hat{Z}^{(1,0)}, \\ \hat{\gamma}^{(2)} &= 6\hat{Z}^{(3,1)} - 4\hat{Z}^{(2,1)}\hat{Z}^{(1,0)} - 2\hat{Z}^{(1,1)}\hat{Z}^{(2,0)}.\end{aligned}\tag{35}$$

The matrices $\hat{Z}^{(1,0)}$, $\hat{Z}^{(1,1)}$, $\hat{Z}^{(2,0)}$ and $\hat{Z}^{(2,1)}$ are found by calculating various one- and two-loop diagrams with a single insertion of Q_1 – Q_6 , $E_1^{(1)}$ – $E_4^{(1)}$ and $E_1^{(2)}$ – $E_4^{(2)}$, whereas the matrix $\hat{Z}^{(3,1)}$ requires the computation of three-loop diagrams with insertions of Q_1 – Q_6 as shown in Figure 5. The pole and finite parts of these one-, two- and three-loop diagrams are evaluated using the method we have described together with Paolo Gambino in detail in [15]: We perform the calculation off-shell in an arbitrary R_ξ gauge which allows us to explicitly check the gauge-parameter independence of the mixing among physical operators. To distinguish between IR and UV divergences we follow [17, 18] and introduce a common mass M for all fields, expanding all loop integrals in inverse powers of M . This makes the calculation of the UV divergences possible even at three loops, as M becomes the only relevant internal scale and three-loop tadpole integrals with a single non-zero mass are known [18, 32]. On the other hand, this procedure requires to take into account insertions of the non-physical operators $N_1^{(1)}$ and $N_1^{(2)}$ – $N_{10}^{(2)}$, as well as of appropriate counterterms of dimension-three and four, some of which explicitly break gauge invariance. A comprehensive discussion of the technical details of the renormalization of the effective theory and the actual calculation of the operator mixing is given in [15].

Having summarized the general formalism and our method, we will now present our results for an arbitrary number of quark flavors denoted by f . For completeness we start with the regularization- and renormalization-scheme independent matrix $\hat{\gamma}^{(0)}$, which is

given by

$$\hat{\gamma}^{(0)} = \begin{pmatrix} -4 & \frac{8}{3} & 0 & -\frac{2}{9} & 0 & 0 \\ 12 & 0 & 0 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & -\frac{52}{3} & 0 & 2 \\ 0 & 0 & -\frac{40}{9} & -\frac{160}{9} + \frac{4}{3}f & \frac{4}{9} & \frac{5}{6} \\ 0 & 0 & 0 & -\frac{256}{3} & 0 & 20 \\ 0 & 0 & -\frac{256}{9} & -\frac{544}{9} + \frac{40}{3}f & \frac{40}{9} & -\frac{2}{3} \end{pmatrix}. \quad (36)$$

While the matrix $\hat{\gamma}^{(0)}$ is renormalization-scheme independent, $\hat{\gamma}^{(1)}$ and $\hat{\gamma}^{(2)}$ are not. In the $\overline{\text{MS}}$ scheme supplemented by the definition of evanescent operators given in Eqs. (23), (24) and (25) we obtain

$$\hat{\gamma}^{(1)} = \begin{pmatrix} -\frac{145}{3} + \frac{16}{9}f & -26 + \frac{40}{27}f & -\frac{1412}{243} & -\frac{1369}{243} & \frac{134}{243} & -\frac{35}{162} \\ -45 + \frac{20}{3}f & -\frac{28}{3} & -\frac{416}{81} & \frac{1280}{81} & \frac{56}{81} & \frac{35}{27} \\ 0 & 0 & -\frac{4468}{81} & -\frac{29129}{81} - \frac{52}{9}f & \frac{400}{81} & \frac{3493}{108} - \frac{2}{9}f \\ 0 & 0 & -\frac{13678}{243} + \frac{368}{81}f & -\frac{79409}{243} + \frac{1334}{81}f & \frac{509}{486} - \frac{8}{81}f & \frac{13499}{648} - \frac{5}{27}f \\ 0 & 0 & -\frac{244480}{81} - \frac{160}{9}f & -\frac{29648}{81} - \frac{2200}{9}f & \frac{23116}{81} + \frac{16}{9}f & \frac{3886}{27} + \frac{148}{9}f \\ 0 & 0 & \frac{77600}{243} - \frac{1264}{81}f & -\frac{28808}{243} + \frac{164}{81}f & -\frac{20324}{243} + \frac{400}{81}f & -\frac{21211}{162} + \frac{622}{27}f \end{pmatrix}, \quad (37)$$

and

$$\hat{\gamma}^{(2)} = \begin{pmatrix} -\frac{1927}{2} + \frac{257}{9}f + \frac{40}{9}f^2 + (224 + \frac{160}{3}f)\zeta_3 & \frac{475}{9} + \frac{362}{27}f - \frac{40}{27}f^2 - (\frac{896}{3} + \frac{320}{9}f)\zeta_3 \\ \frac{307}{2} + \frac{361}{3}f - \frac{20}{3}f^2 - (1344 + 160f)\zeta_3 & \frac{1298}{3} - \frac{76}{3}f - 224\zeta_3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (38)$$

$$\begin{pmatrix} \frac{269107}{13122} - \frac{2288}{729}f - \frac{1360}{81}\zeta_3 & -\frac{2425817}{13122} + \frac{30815}{4374}f - \frac{776}{81}\zeta_3 \\ \frac{69797}{2187} + \frac{904}{243}f + \frac{2720}{27}\zeta_3 & \frac{1457549}{8748} - \frac{22067}{729}f - \frac{2768}{27}\zeta_3 \\ -\frac{4203068}{2187} + \frac{14012}{243}f - \frac{608}{27}\zeta_3 & -\frac{18422762}{2187} + \frac{888605}{2916}f + \frac{272}{27}f^2 + (\frac{39824}{27} + 160f)\zeta_3 \\ -\frac{5875184}{6561} + \frac{217892}{2187}f + \frac{472}{81}f^2 + (\frac{27520}{81} + \frac{1360}{9}f)\zeta_3 & -\frac{70274587}{13122} + \frac{8860733}{17496}f - \frac{4010}{729}f^2 + (\frac{16592}{81} + \frac{2512}{27}f)\zeta_3 \\ -\frac{194951552}{2187} + \frac{358672}{81}f - \frac{2144}{81}f^2 + \frac{87040}{27}\zeta_3 & -\frac{130500332}{2187} - \frac{2949616}{729}f + \frac{3088}{27}f^2 + (\frac{238016}{27} + 640f)\zeta_3 \\ \frac{162733912}{6561} - \frac{2535466}{2187}f + \frac{17920}{243}f^2 + (\frac{174208}{81} + \frac{12160}{9}f)\zeta_3 & \frac{13286236}{6561} - \frac{1826023}{4374}f - \frac{159548}{729}f^2 - (\frac{24832}{81} + \frac{9440}{27}f)\zeta_3 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{343783}{52488} + \frac{392}{729}f + \frac{124}{81}\zeta_3 & -\frac{37573}{69984} + \frac{35}{972}f + \frac{100}{27}\zeta_3 \\ -\frac{37889}{8748} - \frac{28}{243}f - \frac{248}{27}\zeta_3 & \frac{366919}{11664} - \frac{35}{162}f - \frac{110}{9}\zeta_3 \\ \frac{674281}{4374} - \frac{1352}{243}f - \frac{496}{27}\zeta_3 & \frac{9284531}{11664} - \frac{2798}{81}f - \frac{26}{27}f^2 - (\frac{1921}{9} + 20f)\zeta_3 \\ \frac{2951809}{52488} - \frac{31175}{8748}f - \frac{52}{81}f^2 - (\frac{3154}{81} + \frac{136}{9}f)\zeta_3 & \frac{3227801}{8748} - \frac{105293}{11664}f - \frac{65}{54}f^2 + (\frac{200}{27} - \frac{220}{9}f)\zeta_3 \\ \frac{14732222}{2187} - \frac{27428}{81}f + \frac{272}{81}f^2 - \frac{13984}{27}\zeta_3 & \frac{16521659}{2916} + \frac{8081}{54}f - \frac{316}{27}f^2 - (\frac{22420}{9} + 200f)\zeta_3 \\ -\frac{22191107}{13122} + \frac{395783}{4374}f - \frac{1720}{243}f^2 - (\frac{33832}{81} + \frac{1360}{9}f)\zeta_3 & -\frac{32043361}{8748} + \frac{3353393}{5832}f - \frac{533}{81}f^2 + (\frac{9248}{27} - \frac{1120}{9}f)\zeta_3 \end{pmatrix}.$$

As far as the one- and two-loop mixing of the four-quark operators Q_1 – Q_6 , namely $\hat{\gamma}^{(0)}$ and $\hat{\gamma}^{(1)}$ are concerned, our results agree with those of [11]. Furthermore, they also agree with

the results obtained in [10, 14] after a transformation to the “traditional” operator basis. This will be shown in Section 8 by an explicit calculation. On the other hand, the three-loop mixing of Q_1 – Q_6 described by $\hat{\gamma}^{(2)}$, is entirely new and has never been given before. As it is characteristic for three-loop anomalous dimensions the entries of $\hat{\gamma}^{(2)}$, contain terms proportional to the Riemann zeta function ζ_3 .

6 Renormalization Group Evolution

In this section we shall use the obtained ADM to find the explicit NNLO expressions for the Wilson coefficients

$$C_i(\mu_b) = C_i^{(0)}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} C_i^{(1)}(\mu_b) + \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^2 C_i^{(2)}(\mu_b), \quad (39)$$

with $i = 1$ – 6 , at the low-energy scale $\mu_b = O(m_b)$, which is appropriate for studying non-leptonic B meson decays. Using the general solution of the RGE given in Eq. (6), we arrive at

$$\begin{aligned} C_i^{(0)}(\mu_b) &= \sum_{j=1}^6 c_{0,ij}^{(0)} \eta^{a_j}, \\ C_i^{(1)}(\mu_b) &= \sum_{j=1}^6 \left(c_{0,ij}^{(1)} + c_{1,ij}^{(1)} \eta + e_{1,ij}^{(1)} \eta \tilde{E}_0(x_t) \right) \eta^{a_j}, \\ C_i^{(2)}(\mu_b) &= \sum_{j=1}^6 \left(c_{0,ij}^{(2)} + c_{1,ij}^{(2)} \eta + c_{2,ij}^{(2)} \eta^2 + \left[e_{1,ij}^{(2)} \eta + e_{2,ij}^{(2)} \eta^2 \right] \tilde{E}_0(x_t) \right. \\ &\quad \left. + t_{2,ij}^{(2)} \eta^2 \tilde{T}_0(x_t) + e_{1,ij}^{(1)} \eta^2 \tilde{E}_1(x_t) + g_{2,ij}^{(2)} \eta^2 \tilde{G}_1(x_t) \right) \eta^{a_j}, \end{aligned} \quad (40)$$

where $\eta = \alpha_s(M_W)/\alpha_s(\mu_b)$ and

$$\vec{a}^T = \left(\frac{6}{23} \quad -\frac{12}{23} \quad 0.4086 \quad -0.4230 \quad -0.8994 \quad 0.1456 \right), \quad (41)$$

$$\hat{c}_0^{(0)} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{63} & -\frac{1}{27} & -0.0659 & 0.0595 & -0.0218 & 0.0335 \\ \frac{1}{21} & \frac{1}{9} & 0.0237 & -0.0173 & -0.1336 & -0.0316 \\ -\frac{1}{126} & \frac{1}{108} & 0.0094 & -0.0100 & 0.0010 & -0.0017 \\ -\frac{1}{84} & -\frac{1}{36} & 0.0108 & 0.0163 & 0.0103 & 0.0023 \end{pmatrix}, \quad (42)$$

$$\hat{c}_0^{(1)} = \begin{pmatrix} 5.9606 & 1.0951 & 0 & 0 & 0 & 0 \\ 1.9737 & -1.3650 & 0 & 0 & 0 & 0 \\ -0.5409 & 1.6332 & 1.6406 & -1.6702 & -0.2576 & -0.2250 \\ 2.2203 & 2.0265 & -4.1830 & -0.7135 & -1.8215 & 0.7996 \\ 0.0400 & -0.1861 & -0.1669 & 0.1887 & 0.0201 & 0.0304 \\ -0.2614 & -0.1918 & 0.4197 & 0.0295 & 0.1474 & -0.0640 \end{pmatrix}, \quad (43)$$

$$\hat{c}_1^{(1)} = \begin{pmatrix} 2.0394 & 5.9049 & 0 & 0 & 0 & 0 \\ 1.3596 & -1.9683 & 0 & 0 & 0 & 0 \\ 0.0647 & 0.2187 & -0.4268 & -0.5165 & 0.2832 & -0.2034 \\ 0.0971 & -0.6561 & 0.1534 & 0.1500 & 1.7355 & 0.1916 \\ -0.0162 & -0.0547 & 0.0606 & 0.0865 & -0.0128 & 0.0103 \\ -0.0243 & 0.1640 & 0.0700 & -0.1412 & -0.1339 & -0.0140 \end{pmatrix}, \quad (44)$$

$$\hat{e}_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1933 & 0.1579 & 0.1428 & -0.1074 \\ 0 & 0 & 0.0695 & -0.0459 & 0.8752 & 0.1012 \\ 0 & 0 & 0.0274 & -0.0264 & -0.0064 & 0.0055 \\ 0 & 0 & 0.0317 & 0.0432 & -0.0675 & -0.0074 \end{pmatrix}, \quad (45)$$

$$\hat{c}_0^{(2)} = \begin{pmatrix} 56.4723 & 22.2650 & 0 & 0 & 0 & 0 \\ 14.7825 & -11.7987 & 0 & 0 & 0 & 0 \\ 1.9905 & 19.2386 & -24.6846 & -12.9233 & -4.0085 & 2.0820 \\ 8.1141 & 42.7264 & -11.7014 & -35.4784 & -14.1041 & 4.9828 \\ -0.3660 & -1.2588 & 2.7564 & 0.6168 & 0.2854 & -0.2620 \\ -2.3243 & -3.5577 & 2.9357 & 2.4965 & 1.5568 & -0.4249 \end{pmatrix}, \quad (46)$$

$$\hat{c}_1^{(2)} = \begin{pmatrix} 12.1560 & -6.4667 & 0 & 0 & 0 & 0 \\ 4.0252 & 8.0604 & 0 & 0 & 0 & 0 \\ -1.1032 & -9.6435 & 10.6219 & 14.5052 & 3.3472 & 1.3651 \\ 4.5281 & -11.9660 & -27.0825 & 6.1964 & 23.6695 & -4.8514 \\ 0.0816 & 1.0987 & -1.0803 & -1.6385 & -0.2612 & -0.1847 \\ -0.5332 & 1.1326 & 2.7171 & -0.2564 & -1.9148 & 0.3886 \end{pmatrix}, \quad (47)$$

$$\hat{c}_2^{(2)} = \begin{pmatrix} 32.6228 & 49.8089 & 0 & 0 & 0 & 0 \\ 21.7486 & -16.6030 & 0 & 0 & 0 & 0 \\ 1.0356 & 1.8448 & -0.6250 & -6.6619 & 2.8566 & 0.7622 \\ 1.5535 & -5.5343 & 0.2246 & 1.9350 & 17.5058 & -0.7181 \\ -0.2589 & -0.4612 & 0.0887 & 1.1155 & -0.1290 & -0.0387 \\ -0.3884 & 1.3836 & 0.1026 & -1.8214 & -1.3503 & 0.0524 \end{pmatrix}, \quad (48)$$

$$\hat{e}_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.8111 & -4.4336 & 1.6880 & 0.7207 \\ 0 & 0 & -12.2667 & -1.8940 & 11.9366 & -2.5613 \\ 0 & 0 & -0.4893 & 0.5008 & -0.1317 & -0.0975 \\ 0 & 0 & 1.2307 & 0.0784 & -0.9657 & 0.2051 \end{pmatrix}, \quad (49)$$

$$\hat{e}_2^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.3169 & -0.7444 & 0.4827 & -1.2075 \\ 0 & 0 & 0.4733 & 0.2162 & 2.9582 & 1.1376 \\ 0 & 0 & 0.1869 & 0.1247 & -0.0218 & 0.0613 \\ 0 & 0 & 0.2161 & -0.2035 & -0.2282 & -0.0830 \end{pmatrix}, \quad (50)$$

$$\hat{t}_2^{(2)} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & 0 & 0 & 0 & 0 \\ -\frac{2}{9} & \frac{2}{9} & 0 & 0 & 0 & 0 \\ -\frac{2}{189} & -\frac{2}{81} & 0.0129 & 0.0497 & -0.0092 & -0.0182 \\ -\frac{1}{63} & \frac{2}{27} & -0.0046 & -0.0144 & -0.0562 & 0.0171 \\ \frac{1}{378} & \frac{1}{162} & -0.0018 & -0.0083 & 0.0004 & 0.0009 \\ \frac{1}{252} & -\frac{1}{54} & -0.0021 & 0.0136 & 0.0043 & -0.0012 \end{pmatrix}, \quad (51)$$

$$\hat{g}_2^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7557 & -0.1643 & 0.0861 & 0.3224 \\ 0 & 0 & -0.2716 & 0.0477 & 0.5277 & -0.3038 \\ 0 & 0 & -0.1072 & 0.0275 & -0.0039 & -0.0164 \\ 0 & 0 & -0.1240 & -0.0449 & -0.0407 & 0.0222 \end{pmatrix}. \quad (52)$$

As far as the LO and NLO corrections parameterized by $\hat{c}_0^{(0)}$, $\hat{c}_0^{(1)}$, $\hat{c}_1^{(1)}$ and $\hat{e}_1^{(1)}$ are concerned our results agree perfectly with the findings of [11]. Contrariwise, the resummation of the NNLO logarithms is entirely new, and the corresponding matrices $\hat{c}_0^{(2)}$, $\hat{c}_1^{(2)}$, $\hat{c}_2^{(2)}$, $\hat{e}_1^{(2)}$, $\hat{e}_2^{(2)}$, $\hat{t}_2^{(2)}$ and $\hat{g}_2^{(2)}$ have never been computed before.

7 Renormalization Scheme Dependences

We would now like to elaborate on the question of renormalization scheme dependences in explicit terms, to gain an insight on how they arise beyond the LO, how various quantities transform under a change of scheme, and how the scheme dependences cancels out in physical observables. In this respect we will not only extend the existing NLO results [9, 10] to the NNLO level, but will also discuss the conceptual features related to the

renormalization of α_s that, to the best of our knowledge, have not been studied in the context of the renormalization of effective field theories so far.

It is well-known that beyond LO various quantities such as the ADM or the Wilson coefficients depend on the scheme adopted for the renormalization of the operators present in the effective theory. This scheme dependence arises because the requirement that all UV divergences are removed by a suitable renormalization of parameters, fields as well as operators, does not fix the finite parts of the associated renormalization constants. Indeed, these constants can be defined in different ways corresponding to distinct renormalization schemes, which are always related by a finite renormalization. In the framework of dimensional regularization one example of how such a scheme dependence may occur is the treatment of γ_5 in $n = 4 - 2\epsilon$ dimensions. In this context two well-known choices of scheme are the naive dimensional regularization scheme [33] with γ_5 taken to be fully anticommuting and the 't Hooft-Veltman (HV) scheme [34] which comprises a γ_5 that does not have simple commutation properties with respect to the other Dirac matrices. Another example is the scheme dependence related to the exact form of the physical and evanescent operators chosen in the effective theory. We will discuss the latter issue in great detail in the following section.

In order to show that physical quantities do not depend on the renormalization scheme and on the choice of the operator basis, we have to demonstrate how this dependence cancels out in the matrix elements of the effective hamiltonian introduced in Eq. (1) with $\vec{C}(\mu)$ given by Eq. (15). First, let us denote by $\hat{\gamma}_0^{(i)}$, $\hat{r}_0^{(i)}$ and $\hat{\gamma}_a^{(i)}$, $\hat{r}_a^{(i)}$ with $i = 1, 2$ the results for the ADM and the local parts of the matrix elements, that is, the finite pieces without logarithms of external momenta divided by the renormalization scale squared, obtained in two different renormalization schemes — see Eqs. (5) and (13). Furthermore, let us assume without loss of generality that the first scheme, which we shall call reference scheme hereafter, is distinguished from the other ones by the subsidiary condition $\hat{r}_0^{(1)} = \hat{r}_0^{(2)} = 0$.

It should be clear that for any given scheme a we can always switch to the reference scheme by the following finite renormalization:

$$\hat{Z}_0 = \left[\hat{1} - \frac{\alpha_s(\mu)}{4\pi} \hat{r}_a^{(1)} - \left(\frac{\alpha_s(\mu)}{4\pi} \right)^2 \left(\hat{r}_a^{(2)} - (\hat{r}_a^{(1)})^2 \right) \right] \hat{Z}_a. \quad (53)$$

The corresponding transformations of the $O(\alpha_s^2)$ and $O(\alpha_s^3)$ anomalous dimensions is easily obtained using Eq. (31). At the NLO we reproduce the well-known result [9, 10]

$$\hat{\gamma}_0^{(1)} = \hat{\gamma}_a^{(1)} - [\hat{r}_a^{(1)}, \hat{\gamma}^{(0)}] - 2\beta_0 \hat{r}_a^{(1)}, \quad (54)$$

whereas at the NNLO we find

$$\hat{\gamma}_0^{(2)} = \hat{\gamma}_a^{(2)} - [\hat{r}_a^{(2)}, \hat{\gamma}^{(0)}] - [\hat{r}_a^{(1)}, \hat{\gamma}_a^{(1)}] + \hat{r}_a^{(1)} [\hat{r}_a^{(1)}, \hat{\gamma}^{(0)}] - 4\beta_0 \hat{r}_a^{(2)} - 2\beta_1 \hat{r}_a^{(1)} + 2\beta_0 (\hat{r}_a^{(1)})^2. \quad (55)$$

Similarly, for the transformation of the Wilson coefficients through NNLO we obtain

$$\vec{C}_0(\mu) = \left[\hat{1} + \frac{\alpha_s(\mu)}{4\pi} \hat{r}_a^{(1)} + \left(\frac{\alpha_s(\mu)}{4\pi} \right)^2 \hat{r}_a^{(2)} \right]^T \vec{C}_a(\mu), \quad (56)$$

where again the NLO result is known for quite some time [35]. The relations connecting the ADM and the Wilson coefficients in two different schemes a and b can easily be derived from the above equations.

With Eqs. (54) and (55) at hand, it is now straightforward to show that the matrices $\hat{R}^{(1)}$ and $\hat{R}^{(2)}$ introduced in Eq. (16) are independent of the renormalization scheme and the form of the operators considered. We start from the anomalous dimensions in the reference scheme $\hat{\gamma}_0^{(1)}$ and $\hat{\gamma}_0^{(2)}$. These matrices can be accessed from any arbitrary scheme a using Eqs. (54) and (55). Let us transpose the latter equations and eliminate $\hat{\gamma}_a^{(1)T}$ and $\hat{\gamma}_a^{(2)T}$ by means of Eq. (A.2). Finally, dropping the unnecessary subscript a , we obtain

$$\begin{aligned} \hat{\gamma}_0^{(1)T} &= \frac{\beta_1}{\beta_0} \hat{\gamma}^{(0)T} - \left[\hat{\gamma}^{(0)T}, \hat{R}^{(1)} \right] - 2\beta_0 \hat{R}^{(1)}, \\ \hat{\gamma}_0^{(2)T} &= \frac{\beta_2}{\beta_0} \hat{\gamma}^{(0)T} - \left[\hat{\gamma}^{(0)T}, \hat{R}^{(2)} \right] - \frac{\beta_1}{\beta_0} \left[\hat{\gamma}^{(0)T}, \hat{R}^{(1)} \right] + \left[\hat{\gamma}^{(0)T}, \hat{R}^{(1)} \right] \hat{R}^{(1)} \\ &\quad - 4\beta_0 \hat{R}^{(2)} - 2\beta_1 \hat{R}^{(1)} + 2\beta_0 (\hat{R}^{(1)})^2, \end{aligned} \quad (57)$$

which proves the scheme independence of $\hat{R}^{(1)}$ and $\hat{R}^{(2)}$.

Next, $\vec{A}^{(0)}$, $\vec{A}^{(1)}$ and $\vec{A}^{(2)}$, obtained from the calculation in the full theory, clearly do not depend on the particular choice adopted for the renormalization of operators. In consequence, the factor to the right of $\hat{U}^{(0)}(\mu, \mu_0)$ in $\vec{C}(\mu)$, as given in Eq. (15), which is related to the upper end of the evolution, is independent of the renormalization scheme. The same is true for the LO evolution matrix $\hat{U}^{(0)}(\mu, \mu_0)$. However, $\vec{C}(\mu)$ still depends on the renormalization scheme through $\hat{K}(\mu)$ and consequently on $\hat{J}^{(1)}$ and $\hat{J}^{(2)}$, entering the Wilson coefficients to the left of $\hat{U}^{(0)}(\mu, \mu_0)$. As is evident from Eqs. (7) and (13), this dependence on the lower end of the evolution is canceled by the one of the matrix elements $\langle \vec{Q}^T(\mu) \rangle$. We have therefore explicitly seen that the matrix elements of the effective hamiltonian and the resulting physical amplitudes are scheme independent.

It is important to emphasize that the renormalization scheme dependence discussed above refers to the renormalization of operators only, and has to be distinguished from the renormalization scheme dependence related to a redefinition of the strong coupling constant. In the following we will discuss the latter issue in detail, illustrating which effect a change in the charge renormalization has on the miscellaneous ingredients of the renormalization group improved perturbation theory. Finally, we will also prove that the matrix elements in the effective theory are invariant under a change of coupling constant.

The coupling parameter $\alpha_s(\mu)$ in the new scheme will be denoted by $\alpha'_s(\mu)$ and the finite renormalization relating the two schemes is written as

$$\alpha_s(\mu) = \left[1 + \frac{\alpha'_s(\mu)}{4\pi} c_1 + \left(\frac{\alpha'_s(\mu)}{4\pi} \right)^2 c_2 \right] \alpha'_s(\mu). \quad (58)$$

It is now easy to show that the first two terms in the strong coupling expansion of the QCD β function, that is, β_0 and β_1 , are scheme independent, while the third coefficient, namely β_2 , transforms non-trivially [36]:

$$\beta'_2 = \beta_2 + c_1 \beta_1 - (c_2 - c_1^2) \beta_0. \quad (59)$$

The renormalization constant matrices \hat{Z} and \hat{Z}' of the two different schemes can in general be related through a finite renormalization $\hat{\rho}(g')$ in such a way that

$$\hat{Z} = \hat{\rho}(g') \hat{Z}', \quad \text{with} \quad \hat{\rho}(g') = \hat{1} + \sum_{k=1}^{\infty} \left(\frac{g'^2}{16\pi^2} \right)^k \hat{\rho}^{(k)}. \quad (60)$$

Inserting this into Eq. (31) we obtain for the ADM in the primed scheme

$$\hat{\gamma}'(g') = \hat{\rho}(g')^{-1} \left(\hat{\gamma}(g) + \mu \frac{d}{d\mu} \right) \hat{\rho}(g'). \quad (61)$$

Comparing Eqs. (31), (53), (60) and (61) it should be clear that the NLO counterpart of Eq. (54) corresponding to the latter transformation reads

$$\hat{\gamma}'^{(1)} = \hat{\gamma}^{(1)} - [\hat{\rho}^{(1)}, \hat{\gamma}^{(0)}] - 2\beta_0 \hat{\rho}^{(1)} + c_1 \hat{\gamma}^{(0)}, \quad (62)$$

whereas the NNLO analog of Eq. (55) takes the following form

$$\begin{aligned} \hat{\gamma}'^{(2)} = & \hat{\gamma}^{(2)} - [\hat{\rho}^{(2)}, \hat{\gamma}^{(0)}] - [\hat{\rho}^{(1)}, \hat{\gamma}^{(1)}] + \hat{\rho}^{(1)} [\hat{\rho}^{(1)}, \hat{\gamma}^{(0)}] - 4\beta_0 \hat{\rho}^{(2)} \\ & - 2\beta_1 \hat{\rho}^{(1)} + 2\beta_0 (\hat{\rho}^{(1)})^2 - c_1 [\hat{\rho}^{(1)}, \hat{\gamma}^{(0)}] + c_2 \hat{\gamma}^{(0)} + 2c_1 \hat{\gamma}^{(1)}. \end{aligned} \quad (63)$$

Needless to say, that the finite operator renormalization $\hat{\rho}$ is unambiguously determined by the exact form of the change of charge renormalization parameterized by c_1 and c_2 . Yet, it is not difficult to understand that by making use of the prior established fact that the matrix elements of the effective hamiltonian introduced in Eq. (1) are renormalization scheme independent one can always remove the dependence on $\hat{\rho}^{(1)}$ and $\hat{\rho}^{(2)}$ from Eqs. (62) and (63). In other words, it is always possible to choose a scheme in which the operator renormalization constants remain pure UV poles. In consequence, the explicit relations between the coefficients codifying the finite charge and operator renormalization are not needed to prove the invariance of the effective theory under a redefinition of the strong coupling constant. Therefore we will not give this relations here.

It is easy to see, that the suitable change of renormalization scheme that one has to perform in order to remove the finite renormalization matrices $\hat{\rho}^{(1)}$ and $\hat{\rho}^{(2)}$ in Eqs. (62) and (63), is characterized through

$$\hat{r}_a^{(1)} = -\hat{\rho}^{(1)}, \quad \text{and} \quad \hat{r}_a^{(2)} = -\hat{\rho}^{(2)} + (\hat{\rho}^{(1)})^2, \quad (64)$$

and implemented by Eqs. (54) and (55). While the strong coupling constant is obviously invariant under such a change of renormalization scheme, that is,

$$\alpha_s''(\mu) = \alpha_s'(\mu), \quad (65)$$

the anomalous dimensions beyond LO transform non-trivially. Combining Eq. (60) with the latter transformation we obtain the following NLO anomalous dimensions in the double primed scheme

$$\hat{\gamma}''^{(1)} = \hat{\gamma}^{(1)} + c_1 \hat{\gamma}^{(0)}, \quad (66)$$

whereas the NNLO result reads

$$\hat{\gamma}''^{(2)} = \hat{\gamma}^{(2)} + c_2 \hat{\gamma}^{(0)} + 2c_1 \hat{\gamma}^{(1)}, \quad (67)$$

and the dependence on $\hat{\rho}^{(1)}$ and $\hat{\rho}^{(2)}$ has dropped out from the last two equations as intended.

Next, let us write down the explicit expressions for the matrices $\hat{J}^{(1)}$ and $\hat{J}^{(2)}$ in the double primed scheme. Using Eq. (A.2) and taking into account that the expansion coefficients of the QCD β function in the double primed scheme coincide with those in the primed one, we find up to the NNLO level

$$\begin{aligned} \hat{J}''^{(1)} &= \hat{J}^{(1)} - \frac{c_1}{2\beta_0} \hat{\gamma}^{(0)T}, \\ \hat{J}''^{(2)} &= \hat{J}^{(2)} + c_1 \hat{J}^{(1)} - \frac{c_1}{2\beta_0} \hat{J}^{(1)} \hat{\gamma}^{(0)T} + \frac{c_1^2 - 2c_2}{4\beta_0} \hat{\gamma}^{(0)T} + \frac{c_1^2}{8\beta_0^2} (\hat{\gamma}^{(0)T})^2, \end{aligned} \quad (68)$$

with $\hat{J}^{(1)}$ and $\hat{J}^{(2)}$ defined as in Eq. (10).

Furthermore, in order to separate the coupling scheme from the renormalization scheme dependence, let us assume without loss of generality, that the local parts of the matrix elements in the unprimed scheme fulfill the constraint $\hat{r}^{(1)} = \hat{r}^{(2)} = 0$. From Eqs. (66) and (67) it is then immediately clear that the local parts of the matrix elements in the double primed scheme correspondingly satisfy the relation $r''^{(1)} = r''^{(2)} = 0$. These two subsidiary conditions together with Eq. (68) now imply that the matrices $\hat{R}^{(1)}$ and $\hat{R}^{(2)}$ in the double primed scheme are given by

$$\begin{aligned} \hat{R}''^{(1)} &= \hat{R}^{(1)} - \frac{c_1}{2\beta_0} \hat{\gamma}^{(0)T}, \\ \hat{R}''^{(2)} &= \hat{R}^{(2)} + c_1 \hat{R}^{(1)} - \frac{c_1}{2\beta_0} \hat{R}^{(1)} \hat{\gamma}^{(0)T} + \frac{c_1^2 - 2c_2}{4\beta_0} \hat{\gamma}^{(0)T} + \frac{c_1^2}{8\beta_0^2} (\hat{\gamma}^{(0)T})^2, \end{aligned} \quad (69)$$

with $\hat{R}^{(1)}$ and $\hat{R}^{(2)}$ defined as in Eq. (16).

For the sake of completeness let us remark that only the third coefficient in the expansion of the amplitude calculated in the full theory transforms non-trivially under a redefinition of the strong coupling constant:

$$\vec{A}''^{(2)} = \vec{A}^{(2)} + c_1 \vec{A}^{(1)}. \quad (70)$$

To prove that the effective theory does in fact not depend on the renormalization scheme employed for the strong coupling constant, let us first write down the analog of Eq. (15) in the double primed scheme:

$$\begin{aligned} \vec{C}''(\mu) = \hat{K}''(\mu) \hat{U}''^{(0)}(\mu, \mu_0) & \left(\vec{A}''^{(0)} + \frac{\alpha_s''(\mu_0)}{4\pi} \left[\vec{A}''^{(1)} - \hat{R}''^{(1)} \vec{A}''^{(0)} \right] \right. \\ & \left. + \left(\frac{\alpha_s''(\mu_0)}{4\pi} \right)^2 \left[\vec{A}''^{(2)} - \hat{R}''^{(1)} \vec{A}''^{(1)} - \left(\hat{R}''^{(2)} - (\hat{R}''^{(1)})^2 \right) \vec{A}''^{(0)} \right] \right), \end{aligned} \quad (71)$$

where

$$\hat{K}''(\mu) = \hat{1} + \frac{\alpha_s(\mu)}{4\pi} \hat{J}''^{(1)} + \left(\frac{\alpha_s(\mu)}{4\pi} \right)^2 \left(\hat{J}''^{(2)} - c_1 \hat{J}''^{(1)} \right), \quad (72)$$

and

$$\hat{U}''^{(0)}(\mu, \mu_0) = \hat{V} \text{diag} \left(\frac{\alpha_s''(\mu_0)}{\alpha_s''(\mu)} \right)^{a_i} \hat{V}^{-1}, \quad (73)$$

with \hat{V} and a_i defined as in Eq. (9).

It is now a matter of simply algebra to show that the LO evolution matrix in the double primed scheme is related to the LO evolution matrix in the unprimed scheme by

$$\hat{U}''^{(0)}(\mu, \mu_0) = \hat{L}(\mu) \hat{U}^{(0)}(\mu, \mu_0) \hat{L}^{-1}(\mu_0), \quad (74)$$

where

$$\begin{aligned} \hat{L}(\mu) &= \hat{1} + \frac{\alpha_s(\mu)}{4\pi} \frac{c_1}{2\beta_0} \hat{\gamma}^{(0)T} - \left(\frac{\alpha_s(\mu)}{4\pi} \right)^2 \left[\frac{3c_1^2 - 2c_2}{4\beta_0} \hat{\gamma}^{(0)T} - \frac{c_1^2}{8\beta_0^2} (\hat{\gamma}^{(0)T})^2 \right], \\ \hat{L}^{-1}(\mu_0) &= \hat{1} - \frac{\alpha_s(\mu_0)}{4\pi} \frac{c_1}{2\beta_0} \hat{\gamma}^{(0)T} + \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^2 \left[\frac{3c_1^2 - 2c_2}{4\beta_0} \hat{\gamma}^{(0)T} + \frac{c_1^2}{8\beta_0^2} (\hat{\gamma}^{(0)T})^2 \right]. \end{aligned} \quad (75)$$

Taking into account Eq. (70) and expanding the double primed Wilson coefficient $\vec{C}''(\mu)$ in terms of the unprimed coupling parameter leads to

$$\begin{aligned} \vec{C}''(\mu) = \hat{K}''(\mu) \hat{L}(\mu) \hat{U}^{(0)}(\mu, \mu_0) \hat{L}^{-1}(\mu_0) & \left(\vec{A}^{(0)} + \frac{\alpha_s(\mu_0)}{4\pi} \left[\vec{A}^{(1)} - \hat{R}''^{(1)} \vec{A}^{(0)} \right] \right. \\ & \left. + \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^2 \left[\vec{A}^{(2)} - \hat{R}''^{(1)} \left(\vec{A}^{(1)} - c_1 \vec{A}^{(0)} \right) - \left(\hat{R}''^{(2)} - (\hat{R}''^{(1)})^2 \right) \vec{A}^{(0)} \right] \right). \end{aligned} \quad (76)$$

Inserting Eqs. (68), (69), (72) and (75) into the latter equation we finally obtain

$$\begin{aligned} \vec{C}''(\mu) = \hat{K}(\mu) \hat{U}^{(0)}(\mu, \mu_0) & \left(\vec{A}^{(0)} + \frac{\alpha_s(\mu_0)}{4\pi} \left[\vec{A}^{(1)} - \hat{R}^{(1)} \vec{A}^{(0)} \right] \right. \\ & \left. + \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^2 \left[\vec{A}^{(2)} - \hat{R}^{(1)} \vec{A}^{(1)} - \left(\hat{R}^{(2)} - (\hat{R}^{(1)})^2 \right) \vec{A}^{(0)} \right] \right), \end{aligned} \quad (77)$$

which shows that $\vec{C}''(\mu)$ is nothing but $\vec{C}(\mu)$. Since $\vec{C}(\mu)$ does not depend on the scheme used to renormalize the high scale coupling constant, the same is obviously true for $\vec{C}''(\mu)$. However, $\vec{C}''(\mu)$ still depends on the renormalization scheme through $\hat{K}(\mu)$ and consequently on $\hat{J}^{(1)}$ and $\hat{J}^{(2)}$, entering the Wilson coefficients to the left of $\hat{U}^{(0)}(\mu, \mu_0)$. As is evident from Eqs. (7) and (13), and has already been discussed before, this dependence on the lower end of the evolution is canceled by the one of the matrix elements $\langle \vec{Q}^T(\mu) \rangle$. We have therefore explicitly seen that the matrix elements of the effective hamiltonian and the resulting physical amplitudes are in fact invariant under a change of coupling constant.

8 Change of Operator Basis

In $n = 4$ dimensions, a change of the physical operators is always equivalent to a simple linear transformation

$$\vec{Q}' = \hat{R} \vec{Q}, \quad (78)$$

parameterized by a rotation matrix \hat{R} , which affects the renormalization constants and the ADM in a trivial way:

$$\hat{Z}' = \hat{R} \hat{Z} \hat{R}^{-1}, \quad \text{and} \quad \hat{\gamma}' = \hat{R} \hat{\gamma} \hat{R}^{-1}. \quad (79)$$

In the framework of dimensional regularization, the transformation corresponding to the change of basis turns out to be more complicated, as it generally involves evanescent operators as well. This feature basically reflects the fact that in order to formulate consistently the dimensional regularization of a theory containing fermionic degrees of freedom, the Dirac algebra has to be infinite-dimensional, which implies that evanescent operators are necessary to form a complete basis in $n = 4 - 2\epsilon$ dimensions. In consequence, specifying the evanescent operators is necessary to make precise the definition of the $\overline{\text{MS}}$ scheme in the effective theory beyond LO, as can be seen for instance in Eq. (35). Clearly, EOM-vanishing operators are irrelevant to the present discussion.

As long as the change of basis does not mix physical and evanescent operators, the ADM still changes in a trivial way. In particular, a linear transformation of evanescent operators does not affect the physical ADM at all. However, when the change of basis involves linear combinations of evanescent and physical operators, the situation turns out

to be more complicated [11]. Indeed, as we will explain in a moment, the new ADM is still given by Eq. (79), but the presence of evanescent operators induces a finite renormalization constant for the physical operators in the new basis. In order to restore the standard $\overline{\text{MS}}$ scheme definitions, a change of scheme is therefore required.

Let us first consider a change of basis that consists of adding some evanescent operators to the physical ones,

$$\vec{Q}' = \vec{Q} + \hat{W} \vec{E}, \quad (80)$$

parameterized by the matrix \hat{W} . In this case the new ADM is still given by Eq. (79) because of the absence of mixing of evanescent into physical operators in the original basis. However, after the above transformation, the renormalization matrix corresponding to the physical operators in the new basis will contain a finite, non-vanishing contribution

$$\hat{Z}'_{QQ^{(1,0)}} = \hat{W} \hat{Z}_{EQ^{(1,0)}}, \quad (81)$$

where the subscript Q and E denotes an element of the physical and evanescent operators, respectively. In order to re-impose the standard $\overline{\text{MS}}$ conditions, the latter contribution must be removed by a change of scheme, implemented by Eq. (53).

The situation is very similar for a change of basis that consists of adding multiples of ϵ times physical operators to the evanescent ones

$$\vec{E}' = \vec{E} + \epsilon \hat{U} \vec{Q}, \quad (82)$$

parameterized by the matrix \hat{U} . In this case the ADM is unchanged because of its finiteness. However, the renormalization matrix of the physical operators in the new basis will contain a finite, non-vanishing contribution as well:

$$\hat{Z}'_{QQ^{(1,0)}} = -\hat{Z}_{QE^{(1,1)}} \hat{U}. \quad (83)$$

Needless to say, the above contribution must again be removed by a suitable change of scheme, in order to abide by the standard $\overline{\text{MS}}$ renormalization conditions.

We therefore conclude in full generality that a change of basis in dimensional regularization is equivalent to a rotation plus a change of scheme. If we discount possible μ -dependent rotations of the operator basis, it should be clear from the discussion above that the most general change of basis comprises the three linear transformations of Eqs. (78), (80), and (82), as well as a rotation of the evanescent operators, which will be parameterized by the matrix \hat{M} in what follows. In total we thus have

$$\vec{Q}' = \hat{R} \left(\vec{Q} + \hat{W} \vec{E} \right), \quad \text{and} \quad \vec{E}' = \hat{M} \left(\epsilon \hat{U} \vec{Q} + \left[\hat{1} + \epsilon \hat{U} \hat{W} \right] \vec{E} \right). \quad (84)$$

The corresponding residual finite renormalization can be derived with simple algebra. Up to second order in α_s we find

$$\hat{Z}'_{QQ^{(1,0)}} = \hat{R} \left[\hat{W} \hat{Z}_{EQ^{(1,0)}} - \left(\hat{Z}_{QE^{(1,1)}} + \hat{W} \hat{Z}_{EE^{(1,1)}} - \frac{1}{2} \hat{\gamma}^{(0)} \hat{W} \right) \hat{U} \right] \hat{R}^{-1},$$

$$\begin{aligned}\hat{Z}_{QQ}^{\prime(2,0)} = \hat{R} \left[\hat{W} \hat{Z}_{EQ}^{(2,0)} - \left(\hat{Z}_{QE}^{(2,1)} + \hat{W} \hat{Z}_{EE}^{(2,1)} - \frac{1}{4} \hat{\gamma}^{(1)} \hat{W} - \frac{1}{2} \hat{Z}_{QE}^{(1,1)} \hat{Z}_{EQ}^{(1,0)} \hat{W} \right. \right. \\ \left. \left. - \frac{1}{2} \hat{W} \hat{Z}_{EE}^{(1,1)} \hat{Z}_{EQ}^{(1,0)} \hat{W} - \frac{1}{4} \hat{W} \hat{Z}_{EQ}^{(1,0)} \hat{\gamma}^{(0)} \hat{W} + \frac{1}{2} \beta_0 \hat{W} \hat{Z}_{EQ}^{(1,0)} \hat{W} \right) \hat{U} \right] \hat{R}^{-1}. \quad (85)\end{aligned}$$

With these expressions at hand, it is now straightforward to deduce how the ADM and the initial conditions of the Wilson coefficients transforms under the change of basis as given in Eq. (84). Up to the NNLO we obtain

$$\begin{aligned}\hat{\gamma}^{\prime(0)} &= \hat{R} \hat{\gamma}^{(0)} \hat{R}^{-1}, \\ \hat{\gamma}^{\prime(1)} &= \hat{R} \hat{\gamma}^{(1)} \hat{R}^{-1} - \left[\hat{Z}_{QQ}^{\prime(1,0)}, \hat{\gamma}^{\prime(0)} \right] - 2\beta_0 \hat{Z}_{QQ}^{\prime(1,0)}, \\ \hat{\gamma}^{\prime(2)} &= \hat{R} \hat{\gamma}^{(2)} \hat{R}^{-1} - \left[\hat{Z}_{QQ}^{\prime(2,0)}, \hat{\gamma}^{\prime(0)} \right] - \left[\hat{Z}_{QQ}^{\prime(1,0)}, \hat{\gamma}^{\prime(1)} \right] + \left[\hat{Z}_{QQ}^{\prime(1,0)}, \hat{\gamma}^{\prime(0)} \right] \hat{Z}_{QQ}^{\prime(1,0)} \\ &\quad - 4\beta_0 \hat{Z}_{QQ}^{\prime(2,0)} - 2\beta_1 \hat{Z}_{QQ}^{\prime(1,0)} + 2\beta_0 \left(\hat{Z}_{QQ}^{\prime(1,0)} \right)^2,\end{aligned} \quad (86)$$

and

$$\vec{C}'(M_w) = \left[\hat{1} + \frac{\alpha_s(M_w)}{4\pi} \hat{Z}_{QQ}^{\prime(1,0)} + \left(\frac{\alpha_s(M_w)}{4\pi} \right)^2 \hat{Z}_{QQ}^{\prime(2,0)} \right]^T (\hat{R}^{-1})^T \vec{C}(M_w). \quad (87)$$

For what concerns the NLO parts of Eqs. (85), (86) and (87) our findings resemble the formulas derived in [11], if one takes into account that our definition of $\hat{Z}_{QQ}^{\prime(1,0)}$ differs slightly from the residual finite renormalization matrix used in the latter article. On the other hand, the complete NNLO relations Eqs. (85), (86) and (87) have never been presented before.

After these general considerations, let us discuss in some detail how the anomalous dimensions given in Eqs. (36), (37) and (38) are transformed in going to the “traditional” basis of physical operators [1, 10, 14]

$$\begin{aligned}Q'_1 &= (\bar{s}_L^\alpha \gamma_{\mu_1} c_L^\beta) (\bar{c}_L^\beta \gamma^{\mu_1} b_L^\alpha), \\ Q'_2 &= (\bar{s}_L^\alpha \gamma_{\mu_1} c_L^\alpha) (\bar{c}_L^\beta \gamma^{\mu_1} b_L^\beta), \\ Q'_3 &= (\bar{s}_L^\alpha \gamma_{\mu_1} b_L^\alpha) \sum_q (\bar{q}_L^\beta \gamma^{\mu_1} q_L^\beta), \\ Q'_4 &= (\bar{s}_L^\alpha \gamma_{\mu_1} b_L^\beta) \sum_q (\bar{q}_L^\beta \gamma^{\mu_1} q_L^\alpha), \\ Q'_5 &= (\bar{s}_L^\alpha \gamma_{\mu_1} b_L^\alpha) \sum_q (\bar{q}_R^\beta \gamma^{\mu_1} q_R^\beta), \\ Q'_6 &= (\bar{s}_L^\alpha \gamma_{\mu_1} b_L^\beta) \sum_q (\bar{q}_R^\beta \gamma^{\mu_1} q_R^\alpha).\end{aligned} \quad (88)$$

In the above definitions α and β denote color indices.

The one- and two-loop evanescent operators that accompany the “traditional” basis can be found by imposing the requirements given in [14]. At the one-loop level they are

$$E_1^{\prime(1)} = (\bar{s}_L^\alpha \gamma_{\mu_1 \mu_2 \mu_3} c_L^\beta) (\bar{c}_L^\beta \gamma^{\mu_1 \mu_2 \mu_3} b_L^\alpha) - (16 - 4\epsilon) Q'_1,$$

$$\begin{aligned}
E_2'^{(1)} &= (\bar{s}_L^\alpha \gamma_{\mu_1 \mu_2 \mu_3} c_L^\alpha) (\bar{c}_L^\beta \gamma^{\mu_1 \mu_2 \mu_3} b_L^\beta) - (16 - 4\epsilon) Q_2', \\
E_3'^{(1)} &= (\bar{s}_L^\alpha \gamma_{\mu_1 \mu_2 \mu_3} b_L^\alpha) \sum_q (\bar{q}_L^\beta \gamma^{\mu_1 \mu_2 \mu_3} q_L^\beta) - (16 - 4\epsilon) Q_3', \\
E_4'^{(1)} &= (\bar{s}_L^\alpha \gamma_{\mu_1 \mu_2 \mu_3} b_L^\beta) \sum_q (\bar{q}_L^\beta \gamma^{\mu_1 \mu_2 \mu_3} q_L^\alpha) - (16 - 4\epsilon) Q_4', \\
E_5'^{(1)} &= (\bar{s}_L^\alpha \gamma_{\mu_1 \mu_2 \mu_3} b_L^\alpha) \sum_q (\bar{q}_R^\beta \gamma^{\mu_1 \mu_2 \mu_3} q_R^\beta) - (4 + 4\epsilon) Q_5', \\
E_6'^{(1)} &= (\bar{s}_L^\alpha \gamma_{\mu_1 \mu_2 \mu_3} b_L^\beta) \sum_q (\bar{q}_R^\beta \gamma^{\mu_1 \mu_2 \mu_3} q_R^\alpha) - (4 + 4\epsilon) Q_6'. \tag{89}
\end{aligned}$$

Following the same procedure, we find the following two-loop evanescent operators:

$$\begin{aligned}
E_1'^{(2)} &= (\bar{s}_L^\alpha \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} c_L^\beta) (\bar{c}_L^\beta \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} b_L^\alpha) - (256 - 224\epsilon) Q_1', \\
E_2'^{(2)} &= (\bar{s}_L^\alpha \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} c_L^\alpha) (\bar{c}_L^\beta \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} b_L^\beta) - (256 - 224\epsilon) Q_2', \\
E_3'^{(2)} &= (\bar{s}_L^\alpha \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} b_L^\alpha) \sum_q (\bar{q}_L^\beta \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} q_L^\beta) - (256 - 224\epsilon) Q_3', \\
E_4'^{(2)} &= (\bar{s}_L^\alpha \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} b_L^\beta) \sum_q (\bar{q}_L^\beta \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} q_L^\alpha) - (256 - 224\epsilon) Q_4', \\
E_5'^{(2)} &= (\bar{s}_L^\alpha \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} b_L^\alpha) \sum_q (\bar{q}_R^\beta \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} q_R^\beta) - (16 + 128\epsilon) Q_5', \\
E_6'^{(2)} &= (\bar{s}_L^\alpha \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} b_L^\beta) \sum_q (\bar{q}_R^\beta \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} q_R^\alpha) - (16 + 128\epsilon) Q_6'. \tag{90}
\end{aligned}$$

It turns out that in order to transform the ADM given in Eqs. (36), (37) and (38) from the initial set of operators to the “traditional” basis, we have to introduce four additional one-loop evanescent operators

$$\begin{aligned}
E_5^{(1)} &= (\bar{s}_L \gamma_{\mu_1} b_L) \sum_q (\bar{q} \gamma^{\mu_1} \gamma_5 q) - \frac{5}{3} Q_3 + \frac{1}{6} Q_5, \\
E_6^{(1)} &= (\bar{s}_L \gamma_{\mu_1} T^a b_L) \sum_q (\bar{q} \gamma^{\mu_1} \gamma_5 T^a q) - \frac{5}{3} Q_4 + \frac{1}{6} Q_6, \\
E_7^{(1)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3} b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3} \gamma_5 q) - \frac{32}{3} Q_3 + \frac{5}{3} Q_5, \\
E_8^{(1)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3} T^a b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3} \gamma_5 T^a q) - \frac{32}{3} Q_4 + \frac{5}{3} Q_6, \tag{91}
\end{aligned}$$

as well as four additional two-loop evanescent operators

$$\begin{aligned}
E_5^{(2)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \gamma_5 q) - \frac{320}{3} Q_3 + \frac{68}{3} Q_5, \\
E_6^{(2)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} T^a b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \gamma_5 T^a q) - \frac{320}{3} Q_4 + \frac{68}{3} Q_6, \\
E_7^{(2)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7} b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7} \gamma_5 q) - \frac{4352}{3} Q_3 + \frac{1040}{3} Q_5, \\
E_8^{(2)} &= (\bar{s}_L \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7} T^a b_L) \sum_q (\bar{q} \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7} T^a \gamma_5 q) - \frac{4352}{3} Q_4 + \frac{1040}{3} Q_6. \tag{92}
\end{aligned}$$

It should be clear, that the evanescent operators $E_5^{(1)}-E_8^{(1)}$ and $E_5^{(2)}-E_8^{(2)}$ are not needed as counterterms in the initial basis of operators. However, some linear combinations of them will become parts of either the physical or the evanescent operators in the “traditional” basis through the change of basis given by Eq. (84).

At this point a comment concerning the computation of the renormalization constants involving the insertions of the additional evanescent operators is in order. Transforming the three-loop anomalous dimensions from the initial to the “traditional” basis requires the knowledge of one- and two-loop diagrams with insertions of $E_5^{(1)}-E_8^{(1)}$, which introduces traces with γ_5 into the calculation. In this context we follow [37], and avoid anticommutation of γ_5 in any fermionic line containing an odd number of γ_5 . Moreover, while traces containing an odd number of Dirac matrices and a single γ_5 do not pose any problem as they vanish algebraically, we do not evaluate traces containing an even number of Dirac matrices and a single γ_5 in $n = 4 - 2\epsilon$ dimensions. This brings to life new evanescent operators — we will call them trace evanescent in the following — that can in general be written as a contraction of a suitable Dirac structure with one of the following evanescent tensors

$$\widehat{E}_{\mu_1 \dots \mu_m} = \text{Tr}(\gamma_{\mu_1 \dots \mu_m} \gamma_5) - \widetilde{\text{Tr}}(\gamma_{\mu_1 \dots \mu_m} \gamma_5) , \quad (93)$$

where the four-dimensional traces $\widetilde{\text{Tr}}(\gamma_{\mu_1 \dots \mu_m} \gamma_5)$ can be calculated recursively from the initial value $\widetilde{\text{Tr}}(\gamma_5) = 0$ applying

$$\widetilde{\text{Tr}}(\gamma_{\mu_1 \dots \mu_m} \gamma_5) = \sum_{j=2}^m (-1)^j \tilde{g}^{\mu_1 \mu_j} \widetilde{\text{Tr}}(\gamma_{\mu_2 \dots \mu_{j-1} \mu_{j+1} \dots \mu_m} \gamma_5) - \frac{i}{6} \epsilon^{\mu_1 \nu_1 \nu_2 \nu_3} \widetilde{\text{Tr}}(\gamma_{\mu_2 \dots \mu_m \nu_1 \nu_2 \nu_3}) . \quad (94)$$

Here $\tilde{g}^{\mu_1 \mu_2} \equiv \text{diag}(1, -1, -1, -1)$ denotes the four-dimensional metric tensor, and $\epsilon^{\mu_1 \mu_2 \mu_3 \mu_4}$ is the totally antisymmetric Levi-Civita tensor defined so that $\epsilon^{0123} \equiv 1$. Furthermore, the second trace in the above equation should also be taken in $n = 4$ dimensions. It can thus be computed recursively from $\widetilde{\text{Tr}}(1) \equiv 4$ using

$$\widetilde{\text{Tr}}(\gamma_{\mu_1 \dots \mu_m}) = \sum_{j=2}^m (-1)^j \tilde{g}^{\mu_1 \mu_j} \widetilde{\text{Tr}}(\gamma_{\mu_2 \dots \mu_{j-1} \mu_{j+1} \dots \mu_m}) . \quad (95)$$

Apparently, the trace evanescent operators originating from Eq. (93) have to be treated on the same footing as the regular ones introduced earlier on. The idea of introducing more and more evanescent operators seems to make the use of an naive anticommuting γ_5 in multi-loop calculations involving chiral operators futile. Fortunately, for the problem at hand this is not the case, as it turns out that the one-loop insertions of $E_5^{(1)}$ and $E_6^{(1)}$ needed to find the transformation of the two-loop anomalous dimensions between the initial and the “traditional” basis involve only the trace $\text{Tr}(\gamma_{\mu_1 \mu_2} \gamma_5)$, which however is zero. This observation has been made already in [11]. Furthermore, to find the transformation of the three-loop anomalous dimensions matrices, both one-loop insertions of $E_5^{(1)}-E_8^{(1)}$

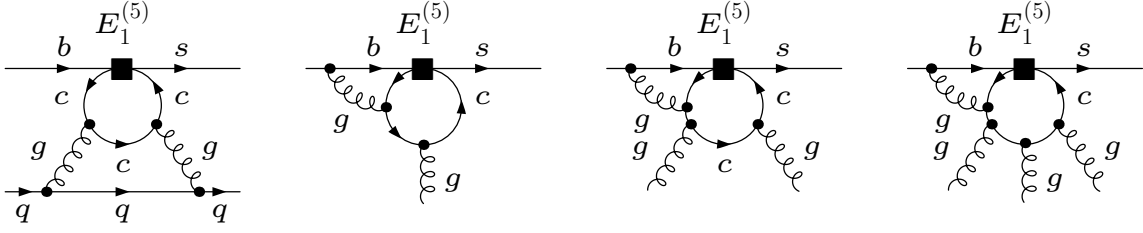


Figure 6: Typical examples of two-loop 1PI diagrams with an insertion of $E_1^{(5)}$ involving non-trivial Dirac traces containing γ_5 .

and two-loop insertions of $E_5^{(1)}$ and $E_6^{(1)}$ are required. Typical examples of non-vanishing two-loop 1PI diagrams are shown in Figure 6. However, also in this case the number of new evanescent structures is rather small, since the necessary operator insertions introduce only the non-trivial trace $\text{Tr}(\gamma_{\mu_1\mu_2\mu_3\mu_4}\gamma_5)$. The complete list of trace evanescent operators relevant to find the transformation of the three-loop anomalous dimensions between the initial and the “traditional” basis is given in Appendix A.2.

If possible any regularization prescription should respect all symmetries of the bare theory, such as gauge and BRST invariance encoded in the Ward and Slavnov-Taylor identities, or Bose symmetry. It is interesting to note that all this requirements are not necessarily fulfilled for an arbitrary choice of trace evanescent tensors. For example, adding any multiple of ϵ times the four-dimensional traces $\widetilde{\text{Tr}}(\gamma_{\mu_1\dots\mu_m}\gamma_5)$ to the right hand side of Eq. (93), in general, spoils the usual Ward and Slavnov-Taylor identities as well as the Bose symmetry of some of the resulting 1PI Green’s functions even after the correct subtraction of all subdivergences. In order to verify that neither problem arises with the definition of trace evanescent operators adopted in Eqs. (A.4) and (A.5), we have calculated the pole and finite parts of the off-shell $b \rightarrow sq\bar{q}$, $b \rightarrow s$, $b \rightarrow sg$, $b \rightarrow sgg$ and $b \rightarrow sg\bar{g}g$ matrix elements with one-loop insertions of $E_5^{(1)}-E_8^{(1)}$ and two-loop insertions of $E_5^{(1)}-E_6^{(1)}$, and checked explicitly that *i*) the resulting operator renormalization constants are independent of the external states used in the calculation, and that *ii*) the subtracted 1PI Green’s functions with two and three external gluons are cyclic under the interchange of any two gluons as required by Bose symmetry. Concerning the former issue, let us mention, that in order to decompose the finite parts of the two-loop matrix elements of $E_5^{(1)}-E_6^{(1)}$ corresponding to the $b \rightarrow sgg$ and $b \rightarrow sg\bar{g}g$ transitions, one has to enlarge the off-shell operator basis to contain besides the EOM-vanishing operators $N_1^{(1)}$ and $N_1^{(2)}-N_{10}^{(2)}$ the following four gauge

non-invariant operators of dimension-six:

$$\begin{aligned}
A_1^{(2)} &= \epsilon_{\mu_1\mu_2\mu_3\mu_4} G^{a\mu_2} \partial^{\mu_3} G^{a\mu_4} (\bar{s}_L \gamma^{\mu_1} b_L), \\
A_2^{(2)} &= \epsilon_{\mu_1\mu_2\mu_3\mu_4} d^{abc} G^{b\mu_2} \partial^{\mu_3} G^{c\mu_4} (\bar{s}_L \gamma^{\mu_1} T^a b_L), \\
A_3^{(2)} &= \epsilon_{\mu_1\mu_2\mu_3\mu_4} f^{abc} G^{a\mu_2} G^{b\mu_3} G^{c\mu_4} (\bar{s}_L \gamma^{\mu_1} b_L), \\
A_4^{(2)} &= \epsilon_{\mu_1\mu_2\mu_3\mu_4} (d^{abe} f^{cde} + d^{ace} f^{dbe} + d^{ade} f^{bce}) G^{a\mu_2} G^{b\mu_3} G^{c\mu_4} (\bar{s}_L \gamma^{\mu_1} T^a b_L).
\end{aligned} \tag{96}$$

Of course the finite two-loop renormalization between $E_5^{(1)}-E_6^{(1)}$ and the anomalous two- and three-gluon operators $A_1^{(2)}-A_4^{(2)}$ does not affect the residual finite renormalization matrices $\hat{Z}_{QQ}^{(1,0)}$ and $\hat{Z}_{QQ}^{(2,0)}$ defined in Eq. (85) which are unambiguously fixed by the corresponding one- and two-loop on-shell $b \rightarrow sq\bar{q}$ and $b \rightarrow sg$ matrix elements. In consequence, the presence of the gauge non-invariant operators $A_1^{(2)}-A_4^{(2)}$ leaves the final result for the anomalous dimensions of the four-quark operators Q_1-Q_6 in the “traditional” scheme unaltered up to the three-loop level.

In case the above considerations might not fully convince a suspicious reader that our treatment of traces containing an odd number of Dirac matrices and a single γ_5 is consistent, it may be worthwhile to justify it in another independent way. To exclude all possibility of doubt concerning our regularization scheme, we have computed the renormalization constants involving the insertions of $E_5^{(1)}-E_8^{(1)}$ using the HV definition of γ_5 in $n = 4 - 2\epsilon$ dimensions, and verified explicitly that, for what concerns the non-anomalous operators, the latter results coincide with those obtained by the “dimensional reduction”-like treatment of traces containing γ_5 , implemented by Eqs. (93), (94) and (95). For the sake of completeness, let us also mention, that in order to decompose the finite parts of the two-loop $b \rightarrow sgg$ and $b \rightarrow sg\bar{g}g$ matrix elements with insertions of $E_5^{(1)}$ and $E_6^{(1)}$ computed in the HV scheme, not four but eight anomalous operators are required. Of course, the appearance of these additional gauge non-invariant operators, which correspond to $A_1^{(2)}-A_4^{(2)}$ with right instead of left chiral quark fields, does not alter the final result for the three-loop $O(\alpha_s^3)$ anomalous dimensions in the “traditional” scheme.

As regards the relevance of the trace evanescent operators defined in Eqs. (A.4) and (A.5), let us finally point out, that it would be incorrect to conclude that they are only needed to find the transformation of the three-loop anomalous dimensions matrices between the initial and the “traditional” basis of operators. In fact, their role in the “traditional” basis is the same as the one of the evanescent operators introduced in Eq. (90), as they, together with the latter ones, make precise the definition of the renormalization scheme in that basis at the NNLO. As should be clear from what has been said in the last section on the cancellation of scheme dependences in general, the corresponding scheme dependence is canceled by the one of the two-loop $O(\alpha_s^2)$ matrix elements, which have to be calculated using exactly the same definition of trace evanescent operators.

The renormalization constant matrices entering Eq. (85) are found from one- and two-loop matrix elements of physical and evanescent operators. We give the relevant ones, as

well as the matrices characterizing the change of basis in Appendix A.3. Our final results for the residual finite renormalization read

$$\hat{Z}_{QQ}'^{(1,0)} = \begin{pmatrix} -\frac{7}{3} & -1 & 0 & 0 & 0 & 0 \\ -2 & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{178}{27} & -\frac{34}{9} & -\frac{164}{27} & \frac{20}{9} \\ 0 & 0 & 1 - \frac{1}{9}f & -\frac{25}{3} + \frac{1}{3}f & -2 - \frac{1}{9}f & 6 + \frac{1}{3}f \\ 0 & 0 & -\frac{160}{27} & \frac{16}{9} & \frac{146}{27} & -\frac{2}{9} \\ 0 & 0 & -2 + \frac{1}{9}f & 6 - \frac{1}{3}f & 3 + \frac{1}{9}f & -\frac{11}{3} - \frac{1}{3}f \end{pmatrix}, \quad (97)$$

and

$$\hat{Z}_{QQ}'^{(2,0)} = \begin{pmatrix} -\frac{200}{9} + \frac{7}{54}f & \frac{68}{3} + \frac{1}{18}f & -\frac{4}{3} & 0 & \frac{4}{3} & 0 \\ -\frac{7}{4} + \frac{1}{9}f & \frac{397}{36} - \frac{1}{27}f & -\frac{77}{162} & -\frac{35}{54} & \frac{77}{162} & \frac{35}{54} \\ 0 & 0 & \frac{57253}{2916} - \frac{23}{108}f & -\frac{74029}{972} - \frac{11}{12}f & -\frac{23768}{729} + \frac{53}{108}f & -\frac{10972}{243} - \frac{5}{4}f \\ 0 & 0 & \frac{4541}{54} - \frac{1213}{972}f & -\frac{165}{2} + \frac{493}{324}f & \frac{1105}{27} - \frac{113}{486}f & \frac{95}{9} + \frac{305}{162}f \\ 0 & 0 & -\frac{6967}{729} + \frac{19}{108}f & \frac{17767}{243} + \frac{37}{36}f & \frac{42199}{1458} - \frac{49}{108}f & \frac{27677}{486} + \frac{41}{36}f \\ 0 & 0 & -\frac{3461}{54} + \frac{1333}{972}f & \frac{1085}{18} - \frac{421}{324}f & -\frac{1375}{27} + \frac{53}{486}f & \frac{43}{3} - \frac{341}{162}f \end{pmatrix}. \quad (98)$$

While the one-loop residual finite renormalization $\hat{Z}_{QQ}'^{(1,0)}$ is independent on the choice of trace evanescent operators, the f parts of the two-loop residual finite renormalization $\hat{Z}_{QQ}'^{(2,0)}$ relating the QCD penguin operators in the unprimed and primed scheme are not. The f parts given in Eq. (98) correspond to the specific choice of trace evanescent operators adopted in Eqs. (A.4) and (A.5).

With these expressions at hand, it is now a matter of simple algebra to find the ADM and the initial conditions of the Wilson coefficients in the “traditional” basis. Using Eqs. (36), (37), (38), (86), (97) and (98) we obtain for the regularization- and renormalization scheme independent one-loop $O(\alpha_s)$ anomalous dimensions matrix:

$$\hat{\gamma}'^{(0)} = \begin{pmatrix} -2 & 6 & 0 & 0 & 0 & 0 \\ 6 & -2 & -\frac{2}{9} & \frac{2}{3} & -\frac{2}{9} & \frac{2}{3} \\ 0 & 0 & -\frac{22}{9} & \frac{22}{3} & -\frac{4}{9} & \frac{4}{3} \\ 0 & 0 & 6 - \frac{2}{9}f & -2 + \frac{2}{3}f & -\frac{2}{9}f & \frac{2}{3}f \\ 0 & 0 & 0 & 0 & 2 & -6 \\ 0 & 0 & -\frac{2}{9}f & \frac{2}{3}f & -\frac{2}{9}f & -16 + \frac{2}{3}f \end{pmatrix}. \quad (99)$$

While the matrix $\hat{\gamma}'^{(0)}$ is renormalization-scheme independent, $\hat{\gamma}'^{(1)}$ and $\hat{\gamma}'^{(2)}$ are not. In the $\overline{\text{MS}}$ scheme supplemented by the definition of evanescent operators given in Eqs. (89) and (90) we obtain for the two-loop $O(\alpha_s^2)$ anomalous dimensions matrix:

$$\hat{\gamma}'^{(1)} = \begin{pmatrix} -\frac{21}{2} - \frac{2}{9}f & \frac{7}{2} + \frac{2}{3}f & \frac{79}{9} & -\frac{7}{3} & -\frac{65}{9} & -\frac{7}{3} \\ \frac{7}{2} + \frac{2}{3}f & -\frac{21}{2} - \frac{2}{9}f & -\frac{202}{243} & \frac{1354}{81} & -\frac{1192}{243} & \frac{904}{81} \\ 0 & 0 & -\frac{5911}{486} + \frac{71}{9}f & \frac{5983}{162} + \frac{1}{3}f & -\frac{2384}{243} - \frac{71}{9}f & \frac{1808}{81} - \frac{1}{3}f \\ 0 & 0 & \frac{379}{18} + \frac{56}{243}f & -\frac{91}{6} + \frac{808}{81}f & -\frac{130}{9} - \frac{502}{243}f & -\frac{14}{3} + \frac{646}{81}f \\ 0 & 0 & -\frac{61}{9}f & -\frac{11}{3}f & \frac{71}{3} + \frac{61}{9}f & -99 + \frac{11}{3}f \\ 0 & 0 & -\frac{682}{243}f & \frac{106}{81}f & -\frac{225}{2} + \frac{1676}{243}f & -\frac{1343}{6} + \frac{1348}{81}f \end{pmatrix}. \quad (100)$$

The f and f^2 parts of the matrix $\hat{\gamma}'^{(2)}$ describing the mixing of the QCD penguin operators do in addition depend on the definition of trace evanescent operators given in Eqs. (A.4) and (A.5). In the corresponding scheme we find for the three-loop $O(\alpha_s^3)$ anomalous dimensions matrix:

$$\hat{\gamma}'^{(2)} = \begin{pmatrix} \frac{19859}{36} - \frac{1543}{18}f + \frac{298}{81}f^2 + \frac{80}{3}f\zeta_3 & -\frac{9}{4} + \frac{605}{18}f - \frac{106}{27}f^2 - (672+80f)\zeta_3 \\ \frac{4741}{12} - \frac{11}{2}f - \frac{82}{27}f^2 - (672+80f)\zeta_3 & -\frac{1165}{36} - \frac{7}{18}f + \frac{82}{81}f^2 + \frac{80}{3}f\zeta_3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (101)$$

$$\begin{pmatrix} \frac{58231}{1944} + \frac{595}{81}f & -\frac{77239}{648} + \frac{53}{27}f \\ -\frac{3664721}{52488} + \frac{23831}{4374}f + \frac{280}{81}\zeta_3 & \frac{4033865}{17496} - \frac{23111}{1458}f - \frac{4024}{27}\zeta_3 \\ \frac{22475861}{13122} + \frac{1025695}{17496}f + \frac{79}{81}f^2 + \left(\frac{560}{81} + \frac{80}{3}f\right)\zeta_3 & \frac{12604483}{4374} - \frac{1382815}{5832}f - \frac{7}{27}f^2 - \left(\frac{26192}{27} + 80f\right)\zeta_3 \\ -\frac{146039}{81} + \frac{8338543}{52488}f - \frac{15961}{4374}f^2 - (672 + \frac{6200}{81}f)\zeta_3 & \frac{31679}{27} - \frac{2583223}{17496}f - \frac{6359}{1458}f^2 - \frac{3304}{81}f\zeta_3 \\ -\frac{100832}{81} - \frac{17705}{486}f - \frac{1}{27}f^2 & -\frac{60448}{27} + \frac{25841}{81}f - \frac{23}{9}f^2 \\ \frac{552928}{243} - \frac{10779689}{52488}f + \frac{18005}{4374}f^2 + \frac{280}{81}f\zeta_3 & -\frac{128992}{81} + \frac{7661297}{17496}f - \frac{31109}{1458}f^2 - \frac{5464}{27}f\zeta_3 \\ -\frac{261287}{1944} + \frac{163}{81}f + \frac{20}{3}\zeta_3 & -\frac{77401}{648} + \frac{53}{27}f - 20\zeta_3 \\ -\frac{2612539}{26244} + \frac{38285}{4374}f + \frac{7228}{81}\zeta_3 & \frac{1722187}{8748} - \frac{16541}{1458}f - \frac{2044}{27}\zeta_3 \\ -\frac{33248683}{13122} + \frac{1049149}{17496}f - \frac{49}{27}f^2 + \left(\frac{14456}{81} + \frac{20}{3}f\right)\zeta_3 & \frac{1098811}{4374} - \frac{266077}{5832}f + \frac{25}{9}f^2 - \left(\frac{4088}{27} + 20f\right)\zeta_3 \\ \frac{437203}{324} - \frac{1360099}{26244}f + \frac{13289}{4374}f^2 + \left(\frac{40}{3} + \frac{7228}{81}f\right)\zeta_3 & -\frac{199123}{108} + \frac{1627075}{8748}f - \frac{6233}{1458}f^2 - \left(40 + \frac{2044}{27}f\right)\zeta_3 \\ \frac{1826987}{648} - \frac{113413}{972}f + \frac{83}{81}f^2 - (378+20f)\zeta_3 & -\frac{472667}{216} + \frac{88933}{324}f - \frac{11}{27}f^2 + (462+60f)\zeta_3 \\ -\frac{2091127}{1944} + \frac{509723}{26244}f + \frac{27815}{4374}f^2 + \left(210 + \frac{7228}{81}f\right)\zeta_3 & -\frac{1946849}{648} + \frac{5194621}{8748}f - \frac{9815}{1458}f^2 + \left(378 + \frac{2276}{27}f\right)\zeta_3 \end{pmatrix}.$$

As far as the one- and two-loop self-mixing of the four-quark operators Q'_1 – Q'_6 , namely $\hat{\gamma}'^{(0)}$ and $\hat{\gamma}'^{(1)}$ are concerned, our findings agree perfectly with the results of the direct computations [10, 14]. Furthermore, they also agree with the results presented in [11], which were obtained by performing a change of scheme from the unprimed set of operators to the primed one. On the other hand, the three-loop self-mixing of Q'_1 – Q'_6 described by $\hat{\gamma}'^{(2)}$, is entirely new.

Similarly, employing Eqs. (26), (87), (97) and (98) we find for the initial conditions of the Wilson coefficients in the “traditional” basis up to $O(\alpha_s^2)$:

$$C'_1(M_W) = \frac{11}{2} \frac{\alpha_s(M_W)}{4\pi} + \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 \left(\frac{2005}{48} + \frac{17}{6}\pi^2 - \frac{1}{2}\tilde{T}_0(x_t) \right),$$

$$C'_2(M_W) = 1 - \frac{11}{6} \frac{\alpha_s(M_W)}{4\pi} - \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 \left(\frac{1405}{144} - \frac{7}{18}\pi^2 - \frac{1}{6}\tilde{T}_0(x_t) \right),$$

$$C'_3(M_W) = -\frac{1}{6} \frac{\alpha_s(M_W)}{4\pi} \tilde{E}_0(x_t) + \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 \left(\frac{539}{810} + \frac{77}{90}\tilde{E}_0(x_t) - \frac{1}{6}\tilde{E}_1(x_t) - \frac{1}{10}\tilde{G}_1(x_t) \right),$$

$$\begin{aligned}
C'_4(M_W) &= \frac{1}{2} \frac{\alpha_s(M_W)}{4\pi} \tilde{E}_0(x_t) + \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 \left(\frac{49}{54} + \frac{7}{6} \tilde{E}_0(x_t) + \frac{1}{2} \tilde{E}_1(x_t) - \frac{3}{2} \tilde{G}_1(x_t) \right), \\
C'_5(M_W) &= -\frac{1}{6} \frac{\alpha_s(M_W)}{4\pi} \tilde{E}_0(x_t) + \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 \left(\frac{308}{405} + \frac{44}{45} \tilde{E}_0(x_t) - \frac{1}{6} \tilde{E}_1(x_t) + \frac{29}{40} \tilde{G}_1(x_t) \right), \\
C'_6(M_W) &= \frac{1}{2} \frac{\alpha_s(M_W)}{4\pi} \tilde{E}_0(x_t) + \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 \left(\frac{28}{27} + \frac{4}{3} \tilde{E}_0(x_t) + \frac{1}{2} \tilde{E}_1(x_t) - \frac{3}{8} \tilde{G}_1(x_t) \right).
\end{aligned} \tag{102}$$

Whereas the two-loop $O(\alpha_s^2)$ corrections to the initial conditions $C'_1(M_W)$ – $C'_6(M_W)$ are entirely new, our findings for the one-loop $O(\alpha_s)$ corrections agree again perfectly with all preceding calculations [10, 11, 14]. Finally, let us mention that the $O(\alpha_s^2)$ corrections to the initial conditions $C'_1(M_W)$ – $C'_6(M_W)$ do not depend on the special choice of trace evanescent operators made in Eqs. (A.4) and (A.5). The NNLO analytic formulas for the low-energy Wilson coefficients relevant for non-leptonic B meson decays in the “traditional” scheme can be found in Appendix A.4.

9 Summary

In this paper we have extended the SM analysis of the effective hamiltonian for non-leptonic $|\Delta F| = 1$ decays to the NNLO. The main ingredient of this generalization is the three-loop ADM describing the mixing of the current-current and QCD penguin operators, which we have computed in an operator basis that allows to consistently use fully anticommuting γ_5 in dimensional regularization to all orders in perturbation theory. The issue of renormalization scheme dependences, their cancellation in physical quantities, and the transformation properties of the ADM and the initial conditions of the Wilson coefficients under a change of scheme has been discussed thoroughly. In particular, we have elaborated on the scheme dependence related to the renormalization of the strong coupling constant, a feature that, to the best of our knowledge, has not been studied in the context of the renormalization of effective field theories so far. As a practical application of our general considerations, we have derived the explicit NNLO relation between our and the so-called “traditional” renormalization scheme, which allowed us to calculate indirectly the NNLO ADM and the corresponding matching conditions in the latter scheme. Finally, we have solved the RGE to obtain the analytic expressions for the low-energy Wilson coefficients relevant for non-leptonic B meson decays through NNLO in both schemes.

Acknowledgments

We are grateful to Paolo Gambino for collaboration at an early stage of this project, for numerous comments and discussions later on, and for his careful proofreading of the present manuscript. Furthermore, we are obliged to Mikolaj Misiak for helpful correspondence and

for his critical reading of the final draft of this paper. In addition, we would like to thank Matthias Steinhauser for providing us with an updated version of **MATAD** [38], and Guido Bell and Diego Guadagnoli for pointing out the typos in Eqs. (27) and (37). Finally, a big thank you goes to Xinqiang Li for bringing the typos/mistakes in Eqs. (26), (29), (48), (102), and (A.27) to our attention. M. G. and U. H. appreciate the warm hospitality of the Institute T31 for Theoretical Elementary Particle Physics at the Technical University of Munich where part of this work has been performed. The work of U. H. is supported by the U.S. Department of Energy under contract No. DE-AC02-76CH03000.

Appendix

A.1 Derivation of $\hat{S}^{(1)}$ and $\hat{S}^{(2)}$

In order to derive the explicit expressions for the matrix kernels $\hat{S}^{(1)}$ and $\hat{S}^{(2)}$ as given in Eq. (11), we follow [9, 39] and compute the partial derivative of Eqs. (4) and (6) with respect to g . After some algebra one finds the following differential equation for $\hat{K}(g)$

$$\frac{\partial \hat{K}(g)}{\partial g} + \frac{1}{g} \left[\frac{\hat{\gamma}^{(0)T}}{\beta_0}, \hat{K}(g) \right] = \left(\frac{\hat{\gamma}^T(g)}{\beta(g)} + \frac{1}{g} \frac{\hat{\gamma}^{(0)T}}{\beta_0} \right) \hat{K}(g). \quad (\text{A.1})$$

Inserting Eq. (7) into the last equation we obtain

$$\begin{aligned} \hat{J}^{(1)} + \left[\frac{\hat{\gamma}^{(0)T}}{2\beta_0}, \hat{J}^{(1)} \right] &= -\frac{\hat{\gamma}^{(1)T}}{2\beta_0} + \frac{\beta_1}{2\beta_0^2} \hat{\gamma}^{(0)T}, \\ \hat{J}^{(2)} + \left[\frac{\hat{\gamma}^{(0)T}}{4\beta_0}, \hat{J}^{(2)} \right] &= -\frac{\hat{\gamma}^{(2)T}}{4\beta_0} + \frac{\beta_1}{4\beta_0^2} \hat{\gamma}^{(1)T} + \left(\frac{\beta_2}{4\beta_0^2} - \frac{\beta_1^2}{4\beta_0^3} \right) \hat{\gamma}^{(0)T} \\ &\quad - \left(\frac{\hat{\gamma}^{(1)T}}{4\beta_0} - \frac{\beta_1}{4\beta_0^2} \hat{\gamma}^{(0)T} \right) \hat{J}^{(1)}, \end{aligned} \quad (\text{A.2})$$

for the parts proportional to g and g^3 , respectively. After diagonalizing these equations with the help of Eq. (9) we find

$$\begin{aligned} S_{ij}^{(1)} &= \frac{\beta_1}{\beta_0} a_i \delta_{ij} - \frac{G_{ij}^{(1)}}{2\beta_0(1+a_i-a_j)}, \\ S_{ij}^{(2)} &= \left(\frac{\beta_2}{2\beta_0} - \frac{\beta_1^2}{2\beta_0^2} \right) a_i \delta_{ij} + \sum_k \frac{2\beta_1 a_i \delta_{ik} - G_{ik}^{(1)}}{2\beta_0(2+a_i-a_j)} S_{kj}^{(1)} + \frac{\beta_1 G_{ij}^{(1)} - \beta_0 G_{ij}^{(2)}}{2\beta_0^2(2+a_i-a_j)}, \end{aligned} \quad (\text{A.3})$$

Finally, solving the first equation for $G_{ij}^{(1)}$ and inserting the result into the second equation, one obtains the expression for the elements of $\hat{S}^{(2)}$ as given in Eq. (11).

A.2 Trace Evanescent Operators

In the following we specify the exact form of the so-called trace evanescent operators arising from the one- and two-loop diagrams with insertions of $E_5^{(1)} - E_8^{(1)}$. At the one-loop level the specific structure of only one of them is needed:

$$\widehat{E}_1^{(1)} = (\bar{s}\gamma_{\mu_1\mu_2\mu_3}P_L b) \text{Tr}(\gamma^{\mu_1\mu_2\mu_3\mu_4}\gamma_5) - 24(\bar{s}_L\gamma^{\mu_4}b_L), \quad (\text{A.4})$$

while at the two-loop level we encounter eight additional trace evanescent operators:

$$\begin{aligned} \widehat{E}_1^{(2)} &= (\bar{s}\gamma_{\mu_1}P_L b) \sum_q (\bar{q}\gamma_{\mu_2\mu_3\mu_4}q) \text{Tr}(\gamma^{\mu_1\mu_2\mu_3\mu_4}\gamma_5) - 40(\bar{s}_L\gamma_{\mu_1}b_L) \sum_q (\bar{q}\gamma^{\mu_1}q) \\ &\quad + 4(\bar{s}_L\gamma_{\mu_1\mu_2\mu_3}b_L) \sum_q (\bar{q}\gamma^{\mu_1\mu_2\mu_3}q), \\ \widehat{E}_2^{(2)} &= (\bar{s}\gamma_{\mu_1}P_L\gamma_{\mu_2\mu_3}b) \text{Tr}(\gamma^{\mu_1\mu_2\mu_3\mu_4}\gamma_5) - 24(\bar{s}_L\gamma^{\mu_4}b_L), \\ \widehat{E}_3^{(2)} &= (\bar{s}\gamma_{\mu_1}P_L\gamma_{\mu_2\mu_3}b) \text{Tr}(\gamma^{\mu_1\mu_3\mu_4\mu_5}\gamma_5) + 8\left(\tilde{g}_{\mu_2}^{\mu_4}(\bar{s}_L\gamma^{\mu_5}b_L) - \tilde{g}_{\mu_2}^{\mu_5}(\bar{s}_L\gamma^{\mu_4}b_L)\right), \\ \widehat{E}_4^{(2)} &= (\bar{s}\gamma_{\mu_1}P_L\gamma_{\mu_2}b) \text{Tr}(\gamma^{\mu_1\mu_2\mu_3\mu_4}\gamma_5) + 8\left((\bar{s}_L\gamma^{\mu_3\mu_4}b_R) - \tilde{g}^{\mu_3\mu_4}(\bar{s}_L b_R)\right), \\ \widehat{E}_5^{(2)} &= (\bar{s}\gamma_{\mu_1\mu_2\mu_3}P_L b) \text{Tr}(\gamma^{\mu_2\mu_3\mu_4\mu_5}\gamma_5) - 8\left((\bar{s}_L\gamma_{\mu_1}^{\mu_4\mu_5}b_L) - \tilde{g}^{\mu_4\mu_5}(\bar{s}_L\gamma_{\mu_1}b_L)\right), \\ \widehat{E}_6^{(2)} &= (\bar{s}\gamma_{\mu_1}P_L b) \text{Tr}(\gamma^{\mu_1\mu_2\mu_3\mu_4}\gamma_5) + 4\left((\bar{s}_L\gamma^{\mu_2\mu_3\mu_4}b_L) \right. \\ &\quad \left. - \tilde{g}^{\mu_2\mu_3}(\bar{s}_L\gamma^{\mu_4}b_L) + \tilde{g}^{\mu_2\mu_4}(\bar{s}_L\gamma^{\mu_3}b_L) - \tilde{g}^{\mu_3\mu_4}(\bar{s}_L\gamma^{\mu_2}b_L)\right), \\ \widehat{E}_7^{(2)} &= (\bar{s}\gamma_{\mu_1\mu_2\mu_3}P_L b) \text{Tr}(\gamma^{\mu_1\mu_2\mu_4\mu_5}\gamma_5) + 8\left((\bar{s}_L\gamma_{\mu_3}^{\mu_4\mu_5}b_L) - 2\tilde{g}_{\mu_3}^{\mu_4}(\bar{s}_L\gamma^{\mu_5}b_L) \right. \\ &\quad \left. + 2\tilde{g}_{\mu_3}^{\mu_5}(\bar{s}_L\gamma^{\mu_4}b_L) - \tilde{g}^{\mu_4\mu_5}(\bar{s}_L\gamma_{\mu_3}b_L)\right), \\ \widehat{E}_8^{(2)} &= (\bar{s}\gamma_{\mu_1}P_L\gamma_{\mu_2\mu_3}b) \text{Tr}(\gamma^{\mu_1\mu_2\mu_4\mu_5}\gamma_5) + 8\left((\bar{s}_L\gamma_{\mu_3}^{\mu_4\mu_5}b_L) - 2\tilde{g}_{\mu_3}^{\mu_4}(\bar{s}_L\gamma^{\mu_5}b_L) \right. \\ &\quad \left. + 2\tilde{g}_{\mu_3}^{\mu_5}(\bar{s}_L\gamma^{\mu_4}b_L) - \tilde{g}^{\mu_4\mu_5}(\bar{s}_L\gamma_{\mu_3}b_L)\right), \end{aligned} \quad (\text{A.5})$$

where we have used the definitions $\gamma_{\mu_1\ldots\mu_j}^{\mu_{j+1}\ldots\mu_n} \equiv \gamma_{\mu_1}\ldots\gamma_{\mu_j}\gamma^{\mu_{j+1}}\ldots\gamma^{\mu_n}$ and $P_L \equiv (1 - \gamma_5)/2$. Finally, let us recall two important features of the latter operators. First, their role in the “traditional” basis is the same as the one played by the evanescent operators in Eq. (90), that is, they are needed to define the renormalization scheme in that basis at the NNLO. Second, the f and f^2 parts of the three-loop ADM $\hat{\gamma}^{(2)}$ describing the mixing of the QCD penguin operators in the “traditional” basis depend on the definition of these operators. The above definitions correspond to the “dimensional reduction”-like treatment of traces with an even number of Dirac matrices and a single γ_5 , employing Eqs. (93), (94) and (95).

A.3 Change to the “Traditional” Operator Basis

In order to give the explicit expressions for the matrices \hat{R} , \hat{W} , \hat{U} and \hat{M} characterizing the change to the “traditional” basis, we first have to define the unprimed and primed set of operators according to Eq. (84). The physical and evanescent operators in the initial basis are given by

$$\begin{aligned}\vec{Q}^T &= (Q_1, \dots, Q_6), \\ \vec{E}^T &= (E_1^{(1)}, \dots, E_8^{(1)}, E_1^{(2)}, \dots, E_8^{(2)}),\end{aligned}\tag{A.6}$$

while the “traditional” basis consists of the following two sets of operators:

$$\begin{aligned}\vec{Q}'^T &= (Q'_1, \dots, Q'_6), \\ \vec{E}'^T &= (E_1'^{(1)}, \dots, E_6'^{(1)}, E_1'^{(2)}, \dots, E_6'^{(2)}, E_3^{(2)}, E_4^{(2)}, E_7^{(2)}, E_8^{(2)}).\end{aligned}\tag{A.7}$$

Needless to say, that $E_5^{(1)}-E_8^{(1)}$ and $E_5^{(2)}-E_8^{(2)}$ play the role of extra, in principle unnecessary operators in the initial operator basis. The same is true for $E_3^{(2)}$, $E_4^{(2)}$, $E_7^{(2)}$ and $E_8^{(2)}$ in the “traditional” basis. They are just included for completeness in the above equations. Although the trace evanescent operators introduced in Eqs. (A.4) and (A.5) are needed to make precise the definition of the “traditional” renormalization scheme we refrain to include them in Eq. (A.7), since they influence the transformation between the unprimed and primed set of operators only in an indirect way, that is, through the finite parts of the two-loop renormalization constants of some of the evanescent operators included in the initial basis. This will be explained at the very end of this appendix.

With this definitions at hand, it is just a matter of simple algebra to find the explicit expressions for the matrices \hat{R} , \hat{W} , \hat{U} and \hat{M} . The rotation matrix \hat{R} , which links the physical operators together, is given by

$$\hat{R} = \begin{pmatrix} 2 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{12} & 0 \\ 0 & 0 & -\frac{1}{9} & -\frac{2}{3} & \frac{1}{36} & \frac{1}{6} \\ 0 & 0 & \frac{4}{3} & 0 & -\frac{1}{12} & 0 \\ 0 & 0 & \frac{4}{9} & \frac{8}{3} & -\frac{1}{36} & -\frac{1}{6} \end{pmatrix}.\tag{A.8}$$

The matrix \hat{W} parametrizes a redefinition of the physical operators \vec{Q} by adding some evanescent operators \vec{E} to them. In the case at hand, \hat{W} reads

$$\hat{W} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.\tag{A.9}$$

On the other hand, \hat{U} describes a redefinition of the evanescent operators \vec{E} by adding some multiples of ϵ times physical operators \vec{Q} to them. The relevant matrix \hat{U} takes the following form

$$\hat{U} = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -112 & 0 & 16 & 0 \\ 0 & 0 & 0 & -112 & 0 & 16 \\ 0 & 0 & -\frac{10}{9} & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & -\frac{10}{9} & 0 & \frac{1}{9} \\ 0 & 0 & -\frac{136}{9} & 0 & \frac{10}{9} & 0 \\ 0 & 0 & 0 & -\frac{136}{9} & 0 & \frac{10}{9} \\ 144 & 0 & 0 & 0 & 0 & 0 \\ 0 & 144 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2224}{9} & 0 & \frac{64}{9} & 0 \\ 0 & 0 & 0 & -\frac{2224}{9} & 0 & \frac{64}{9} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.10})$$

Finally, the matrix \hat{M} represents a simple linear transformation of the evanescent operators. In our case we find

$$\hat{M} = \begin{pmatrix} 2 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{8}{3} & 16 & -\frac{1}{6} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{3} & -4 & \frac{1}{6} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 40 & \frac{20}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 128 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 1 & \frac{128}{3} & 256 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 1 & -\frac{8}{3} & -16 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{A.11})$$

Parts of the above matrices have already been given explicitly in [11], where the change of basis from the initial to the “traditional” basis has been performed including the NLO corrections. If we take into account that the definition of $E_5^{(1)}-E_8^{(1)}$ adopted in Eq. (91)

differs slightly from the definition of $E_5^{(1)}-E_8^{(1)}$ used in [11], our results agree with the expressions given in the latter paper.

The renormalization constant matrices entering Eq. (85) are found from one- and two-loop matrix elements of physical and evanescent operators. In the following we will give only the relevant entries of the necessary renormalization constant matrices, denoting elements that do not affect the final results for the residual finite renormalizations introduced in Eq. (85) with a star. For the finite renormalization between evanescent operators \vec{E} and physical operators \vec{Q} we get

$$\hat{Z}_{EQ}^{(1,0)} = \begin{pmatrix} \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & -\frac{1280}{3} & 320 \\ \star & \star & \star & \star & \frac{640}{9} & \frac{1280}{3} \\ 0 & 0 & \frac{160}{9} & -\frac{128}{9} & -\frac{16}{9} & \frac{4}{3} \\ 0 & 0 & -\frac{80}{27} & -\frac{476}{27}-\frac{2}{3}f & \frac{8}{27} & \frac{16}{9} \\ \star & \star & \star & \star & -\frac{160}{9} & \frac{40}{3} \\ \star & \star & \star & \star & \frac{80}{27} & \frac{160}{9} \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & -\frac{98560}{3} & 24640 \\ \star & \star & \star & \star & \frac{49280}{9} & \frac{98560}{3} \\ \star & \star & \star & \star & -\frac{256}{9} & \frac{64}{3} \\ \star & \star & \star & \star & \frac{128}{27} & \frac{256}{9} \\ \star & \star & \star & \star & \frac{154880}{9} & -\frac{38720}{3} \\ \star & \star & \star & \star & -\frac{77440}{27} & -\frac{154880}{9} \end{pmatrix}. \quad (\text{A.12})$$

For the one-loop mixing of physical operators \vec{Q} into evanescent operators \vec{E} we obtain

$$\hat{Z}_{QE}^{(1,1)} = \begin{pmatrix} \frac{5}{12} & \frac{2}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{9} & \frac{5}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.13})$$

The one-loop mixing among evanescent operators \vec{E} reads

$$\hat{Z}_{EE}^{(1,1)} = \begin{pmatrix} \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & \frac{1}{6} & 0 & -10 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{27} & \frac{5}{72} & -\frac{20}{9} & -\frac{26}{3} & \frac{2}{9} & \frac{5}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \end{pmatrix}. \quad (\text{A.14})$$

At the two-loop order we find for the finite renormalization between evanescent operators \vec{E} and physical operators \vec{Q} :

$$\hat{Z}_{EQ}^{(2,0)} = \begin{pmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ 0 & 0 & \frac{21632}{243} - \frac{44}{81}f & \frac{80092}{243} + \frac{143}{27}f & -\frac{266}{243} + \frac{8}{81}f & -\frac{1208}{81} - \frac{2}{27}f \\ 0 & 0 & -\frac{77020}{729} + \frac{46}{243}f & -\frac{114161}{729} - \frac{59}{18}f & \frac{5182}{729} - \frac{1}{243}f & \frac{21019}{972} - \frac{49}{648}f \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{pmatrix}. \quad (\text{A.15})$$

The two-loop mixing of physical operators \vec{Q} into evanescent operators \vec{E} is given by

$$\hat{Z}_{QE}^{(2,1)} = \begin{pmatrix} \frac{1531}{288} - \frac{5}{216}f & -\frac{1}{72} - \frac{1}{81}f & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{384} & -\frac{35}{864} & \star & \star & 0 & 0 & \star & \star \\ \frac{119}{16} - \frac{1}{18}f & \frac{8}{9} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{35}{192} & -\frac{7}{72} & \star & \star & 0 & 0 & \star & \star \\ 0 & 0 & -\frac{7}{72} & -\frac{35}{192} & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & 0 & 0 & \star & \star \\ 0 & 0 & -\frac{35}{864} & \frac{1}{384} & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & 0 & 0 & \star & \star \\ 0 & 0 & \frac{23}{18} & \frac{51}{4} - \frac{1}{18}f & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & 0 & 0 & \star & \star \\ 0 & 0 & \frac{7}{6} - \frac{1}{81}f & \frac{317}{72} - \frac{5}{216}f & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & 0 & 0 & \star & \star \end{pmatrix}. \quad (\text{A.16})$$

Finally, the two-loop mixing among evanescent operators \vec{E} reads

$$\hat{Z}_{EE}^{(2,1)} = \begin{pmatrix} \star\star & \star & \star & \star & \star & \star & \star & \star\star\star\star & \star & \star & \star\star \\ \star\star & \star & \star & \star & \star & \star & \star & \star\star\star\star & \star & \star & \star\star \\ \star\star & \star & \star & \star & \star & \star & \star & \star\star\star\star & \star & \star & \star\star \\ \star\star & \star & \star & \star & \star & \star & \star & \star\star\star\star & \star & \star & \star\star \\ 00 & \frac{145}{216} & \frac{695}{576} - \frac{1}{108}f & -\frac{157}{9} + 4f & -\frac{1319}{12} + \frac{23}{9}f & \frac{17}{6} & \frac{133}{12} - \frac{1}{18}f & 00 & \star\star & -\frac{7}{72} & -\frac{35}{192} \star\star \\ 00 & \frac{1703}{2592} - \frac{1}{486}f & \frac{2035}{1152} - \frac{5}{1296}f & -\frac{743}{54} + \frac{46}{81}f & -\frac{2819}{36} + \frac{223}{108}f & \frac{43}{54} - \frac{1}{81}f & \frac{379}{72} - \frac{5}{216}f & 00 & \star\star & -\frac{35}{864} & \frac{1}{384} \star\star \\ \star\star & \star & \star & \star & \star & \star & \star & \star\star\star\star & \star & \star & \star\star \\ \star\star & \star & \star & \star & \star & \star & \star & \star\star\star\star & \star & \star & \star\star \\ \star\star & \star & \star & \star & \star & \star & \star & \star\star\star\star & \star & \star & \star\star \\ \star\star & \star & \star & \star & \star & \star & \star & \star\star\star\star & \star & \star & \star\star \\ \star\star & \star & \star & \star & \star & \star & \star & \star\star\star\star & \star & \star & \star\star \\ \star\star & \star & \star & \star & \star & \star & \star & \star\star\star\star & \star & \star & \star\star \\ \star\star & \star & \star & \star & \star & \star & \star & \star\star\star\star & \star & \star & \star\star \\ \star\star & \star & \star & \star & \star & \star & \star & \star\star\star\star & \star & \star & \star\star \\ \star\star & \star & \star & \star & \star & \star & \star & \star\star\star\star & \star & \star & \star\star \end{pmatrix}. \quad (\text{A.17})$$

As far as the one-loop renormalization constant matrices are concerned, let us note, that our results agree with the findings of [11], after taking into account that the definition of $E_5^{(1)} - E_8^{(1)}$ adopted in Eq. (91) differs slightly from definition of $E_5^{(1)} - E_8^{(1)}$ used in the latter article. On the other hand, the two-loop renormalization constant matrices involving the insertion of $E_5^{(1)}$ and $E_6^{(1)}$, are entirely new and have to our knowledge never been computed before. Finally, let us mention that while the matrices $\hat{Z}_{QE}^{(1,0)}$, $\hat{Z}_{QQ}^{(1,1)}$, $\hat{Z}_{EE}^{(1,1)}$, $\hat{Z}_{QE}^{(2,1)}$, $\hat{Z}_{EE}^{(2,1)}$ do not depend on the special choice of trace evanescent operators used throughout the calculation, the f parts of $\hat{Z}_{EQ}^{(2,0)}$ do depend on the latter choice. The f parts of $\hat{Z}_{EQ}^{(2,0)}$ as given in Eq. (A.15) correspond to the specific choice of trace evanescent operators adopted in Eqs. (A.4) and (A.5).

A.4 Wilson Coefficients in the “Traditional” Operator Basis

In this appendix we shall use the ADM given in Eqs. (99), (100) and (101) to find the explicit NNLO expressions for the Wilson coefficients in the “traditional” basis:

$$C'_i(\mu_b) = C_i'^{(0)}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} C_i'^{(1)}(\mu_b) + \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^2 C_i'^{(2)}(\mu_b), \quad (\text{A.18})$$

with $i = 1-6$. Using the general solution of the RGE given in Eq. (6), we arrive at

$$\begin{aligned} C_i'^{(0)}(\mu_b) &= \sum_{j=1}^6 c_{0,ij}'^{(0)} \eta^{a'_j}, \\ C_i'^{(1)}(\mu_b) &= \sum_{j=1}^6 \left(c_{0,ij}'^{(1)} + c_{1,ij}'^{(1)} \eta + e_{1,ij}'^{(1)} \eta \tilde{E}_0(x_t) \right) \eta^{a'_j}, \\ C_i'^{(2)}(\mu_b) &= \sum_{j=1}^6 \left(c_{0,ij}'^{(2)} + c_{1,ij}'^{(2)} \eta + c_{2,ij}'^{(2)} \eta^2 + \left[e_{1,ij}'^{(2)} \eta + e_{2,ij}'^{(2)} \eta^2 \right] \tilde{E}_0(x_t) \right. \\ &\quad \left. + t_{2,ij}'^{(2)} \eta^2 \tilde{T}_0(x_t) + e_{1,ij}'^{(1)} \eta^2 \tilde{E}_1(x_t) + g_{2,ij}'^{(2)} \eta^2 \tilde{G}_1(x_t) \right) \eta^{a'_j}, \end{aligned} \quad (\text{A.19})$$

with

$$\vec{a}'^T = \left(\frac{6}{23} \quad -\frac{12}{23} \quad 0.4086 \quad -0.4230 \quad -0.8994 \quad 0.1456 \right), \quad (\text{A.20})$$

$$\hat{c}_0'^{(0)} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{14} & \frac{1}{6} & 0.0510 & -0.1403 & -0.0113 & 0.0054 \\ -\frac{1}{14} & -\frac{1}{6} & 0.0984 & 0.1214 & 0.0156 & 0.0026 \\ 0 & 0 & -0.0397 & 0.0117 & -0.0025 & 0.0304 \\ 0 & 0 & 0.0335 & 0.0239 & -0.0462 & -0.0112 \end{pmatrix}, \quad (\text{A.21})$$

$$\hat{c}_0'^{(1)} = \begin{pmatrix} 0.8136 & 0.7142 & 0 & 0 & 0 & 0 \\ 0.8136 & -0.7142 & 0 & 0 & 0 & 0 \\ -0.0766 & -0.1455 & -0.8848 & 0.4137 & -0.0114 & 0.1722 \\ -0.2353 & -0.0397 & 0.4920 & -0.2758 & 0.0019 & -0.1449 \\ 0.0397 & 0.0926 & 0.7342 & -0.1261 & -0.1209 & -0.1085 \\ -0.1190 & -0.2778 & -0.5544 & 0.1915 & -0.2744 & 0.3568 \end{pmatrix}, \quad (\text{A.22})$$

$$\hat{c}'^{(1)}_1 = \begin{pmatrix} 1.0197 & 2.9524 & 0 & 0 & 0 & 0 \\ 1.0197 & -2.9524 & 0 & 0 & 0 & 0 \\ -0.1457 & -0.9841 & 0.3299 & 1.2188 & 0.1463 & -0.0328 \\ -0.1457 & 0.9841 & 0.6370 & -1.0547 & -0.2032 & -0.0160 \\ 0 & 0 & -0.2568 & -0.1014 & 0.0320 & -0.1847 \\ 0 & 0 & 0.2168 & -0.2074 & 0.6000 & 0.0679 \end{pmatrix}, \quad (\text{A.23})$$

$$\hat{e}'^{(1)}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1494 & -0.3725 & 0.0738 & -0.0173 \\ 0 & 0 & 0.2885 & 0.3224 & -0.1025 & -0.0084 \\ 0 & 0 & -0.1163 & 0.0310 & 0.0162 & -0.0975 \\ 0 & 0 & 0.0982 & 0.0634 & 0.3026 & 0.0358 \end{pmatrix}, \quad (\text{A.24})$$

$$\hat{c}'^{(2)}_0 = \begin{pmatrix} 7.9372 & 23.1398 & 0 & 0 & 0 & 0 \\ 19.9372 & -23.1398 & 0 & 0 & 0 & 0 \\ -1.8694 & -20.8668 & 5.1370 & 15.9499 & 3.1041 & 1.5552 \\ 2.7077 & 3.4331 & -1.0616 & -0.5150 & -1.8471 & 0.6903 \\ -0.1694 & 11.7230 & -5.3883 & -11.6376 & -0.9016 & 0.1902 \\ -7.6029 & -7.4283 & 9.8588 & 1.6456 & 1.9006 & 1.6776 \end{pmatrix}, \quad (\text{A.25})$$

$$\hat{c}'^{(2)}_1 = \begin{pmatrix} 1.6593 & -4.2175 & 0 & 0 & 0 & 0 \\ 1.6593 & 4.2175 & 0 & 0 & 0 & 0 \\ -0.1561 & 0.8591 & -5.7288 & -3.5925 & 0.1475 & -1.0445 \\ -0.4798 & 0.2344 & 3.1857 & 2.3950 & -0.0252 & 0.8789 \\ 0.0809 & -0.5467 & 4.7534 & 1.0956 & 1.5714 & 0.6584 \\ -0.2428 & 1.6402 & -3.5892 & -1.6632 & 3.5653 & -2.1649 \end{pmatrix}, \quad (\text{A.26})$$

$$\hat{c}'^{(2)}_2 = \begin{pmatrix} 16.3114 & 24.9044 & 0 & 0 & 0 & 0 \\ 16.3114 & -24.9044 & 0 & 0 & 0 & 0 \\ -2.3302 & -8.3015 & 0.4831 & 15.7207 & 1.4758 & 0.1230 \\ -2.3302 & 8.3015 & 0.9329 & -13.6035 & -2.0495 & 0.0598 \\ 0 & 0 & -0.3760 & -1.3081 & 0.3232 & 0.6922 \\ 0 & 0 & 0.3174 & -2.6752 & 6.0523 & -0.2543 \end{pmatrix}, \quad (\text{A.27})$$

$$\hat{e}'^{(2)}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2.5948 & 1.0981 & 0.0744 & -0.5514 \\ 0 & 0 & 1.4429 & -0.7320 & -0.0127 & 0.4640 \\ 0 & 0 & 2.1530 & -0.3349 & 0.7925 & 0.3476 \\ 0 & 0 & -1.6257 & 0.5084 & 1.7980 & -1.1429 \end{pmatrix}, \quad (\text{A.28})$$

$$\hat{e}_2^{\prime(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0180 & 1.7567 & 0.2494 & -0.1948 \\ 0 & 0 & 1.9657 & -1.5201 & -0.3463 & -0.0948 \\ 0 & 0 & -0.7924 & -0.1462 & 0.0546 & -1.0965 \\ 0 & 0 & 0.6689 & -0.2989 & 1.0227 & 0.4029 \end{pmatrix}, \quad (\text{A.29})$$

$$\hat{t}_2^{\prime(2)} = \begin{pmatrix} -\frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{42} & \frac{1}{9} & -0.0100 & -0.1172 & -0.0047 & -0.0029 \\ \frac{1}{42} & -\frac{1}{9} & -0.0193 & 0.1015 & 0.0066 & -0.0014 \\ 0 & 0 & 0.0078 & 0.0098 & -0.0010 & -0.0165 \\ 0 & 0 & -0.0066 & 0.0200 & -0.0194 & 0.0061 \end{pmatrix}, \quad (\text{A.30})$$

$$\hat{g}_2^{\prime(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5842 & 0.3877 & 0.0445 & 0.0520 \\ 0 & 0 & -1.1280 & -0.3355 & -0.0618 & 0.0253 \\ 0 & 0 & 0.4547 & -0.0323 & 0.0097 & 0.2928 \\ 0 & 0 & -0.3839 & -0.0660 & 0.1824 & -0.1076 \end{pmatrix}. \quad (\text{A.31})$$

As far as the LO and NLO corrections parameterized by $\hat{c}_0^{\prime(0)}$, $\hat{c}_0^{\prime(1)}$, $\hat{c}_1^{\prime(1)}$ and $\hat{e}_1^{\prime(1)}$ are concerned our results agree perfectly with the findings of [9, 10, 35]. Contrariwise, the resummation of the NNLO logarithms is entirely new, and the corresponding matrices $\hat{c}_0^{\prime(2)}$, $\hat{c}_1^{\prime(2)}$, $\hat{c}_2^{\prime(2)}$, $\hat{e}_1^{\prime(2)}$, $\hat{e}_2^{\prime(2)}$, $\hat{t}_2^{\prime(2)}$ and $\hat{g}_2^{\prime(2)}$ have never been computed before. Finally, let us mention that while the matrices $\hat{c}_1^{\prime(2)}$, $\hat{c}_2^{\prime(2)}$, $\hat{e}_1^{\prime(2)}$, $\hat{e}_2^{\prime(2)}$, $\hat{t}_2^{\prime(2)}$ and $\hat{g}_2^{\prime(2)}$ do not depend on the special choice of trace evanescent operators used throughout the calculation, the entries of $\hat{c}_0^{\prime(2)}$ describing the evolution of the Wilson coefficients of the QCD penguin operators do depend on the latter choice. The numbers given in Eq. (A.25) correspond to the specific choice of trace evanescent operators adopted in Eqs. (A.4) and (A.5). Recalling what has been said earlier on the cancellation of scheme dependences in general, it should be clear, that the associated scheme dependence is canceled by the one of the two-loop $O(\alpha_s^2)$ matrix elements, which have to be calculated using the same definition of trace evanescent operators.

References

- [1] A. J. Buras, arXiv:hep-ph/9806471 and references therein.
- [2] G. Buchalla *et al.*, Nucl. Phys. **B355** (1991) 305.

- [3] E. Gabrielli and G. F. Giudice, Nucl. Phys. **B433** (1995) 3 , Erratum-ibid. **B507** (1997) 549 [arXiv:hep-lat/9407029]; A. J. Buras *et al.*, Nucl. Phys. **B592** (2001) 55 [arXiv:hep-ph/0007313].
- [4] A. J. Buras *et al.*, Nucl. Phys. **B678** (2004) 455 [arXiv:hep-ph/0306158].
- [5] J. Jang *et al.*, Phys. Rev. **D66** (2002) 055006 [arXiv:hep-ph/0010107].
- [6] E. Gabrielli, A. Masiero and L. Silvestrini, Phys. Lett. **B374** (1996) 80 [arXiv:hep-ph/9509379]; F. Gabbiani *et al.*, Nucl. Phys. **B477** (1996) 321 [arXiv:hep-ph/9604387]; Y. Y. Keum, U. Nierste and A. I. Sanda, Phys. Lett. **B457** (1999) 157 [arXiv:hep-ph/9903230]; A. Masiero and H. Murayama, Phys. Rev. Lett. **83** (1999) 907 [arXiv:hep-ph/9903363]; R. Barbieri, R. Contino and A. Strumia, Nucl. Phys. **B578** (2000) 153 [arXiv:hep-ph/9908255]; A. J. Buras *et al.*, Nucl. Phys. **B566** (2000) 3 [arXiv:hep-ph/9908371]; G. Eyal *et al.*, JHEP **9911** (1999) 032 [arXiv:hep-ph/9908382]; A. L. Kagan and M. Neubert, Phys. Rev. Lett. **83** (1999) 4929 [arXiv:hep-ph/9908404].
- [7] J. Agrawal and P. H. Frampton, Nucl. Phys. **B419** (1994) 254; X. G. He and B. H. J. McKellar, Phys. Rev. **D51** (1995) 6484 [arXiv:hep-ph/9405288]; X. G. He, Phys. Lett. **B460** (1999) 405 [arXiv:hep-ph/9903242]; C. S. Huang, W. J. Huo and Y. L. Wu, Phys. Rev. **D64** (2001) 016009 [arXiv:hep-ph/0005227].
- [8] M. K. Gaillard and B. W. Lee, Phys. Rev. **D10** (1974) 897; G. Altarelli and L. Maiani, Phys. Lett. **B52** (1974) 351; M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. **B120** (1977) 316; F. J. Gilman and M. B. Wise, Phys. Rev. **D20** (1979) 2392; B. Guberina and R. D. Peccei, Nucl. Phys. **B163** (1980) 289.
- [9] A. J. Buras *et al.*, Nucl. Phys. **B370** (1992) 69, Addendum-ibid. **B375** (1992) 501.
- [10] M. Ciuchini *et al.*, Nucl. Phys. **B415** (1994) 403 [arXiv:hep-ph/9304257].
- [11] K. G. Chetyrkin, M. Misiak and M. Münz, Nucl. Phys. **B520** (1998) 279 [arXiv:hep-ph/9711280].
- [12] G. Altarelli *et al.*, Nucl. Phys. **B187** (1981) 461.
- [13] A. J. Buras and P. H. Weisz, Nucl. Phys. **B333** (1990) 66.
- [14] A. J. Buras *et al.*, Nucl. Phys. **B400** (1993) 37 [arXiv:hep-ph/9211304].
- [15] P. Gambino, M. Gorbahn and U. Haisch, Nucl. Phys. **B673** (2003) 238 [arXiv:hep-ph/0306079].
- [16] C. Bobeth, M. Misiak and J. Urban, Nucl. Phys. **B574** (2000) 291 [arXiv:hep-ph/9910220].

- [17] M. Misiak and M. Münz, Phys. Lett. **B344** (1995) 308 [arXiv:hep-ph/9409454].
- [18] K. G. Chetyrkin, M. Misiak and M. Münz, Nucl. Phys. **B518** (1998) 473 [arXiv:hep-ph/9711266].
- [19] A. J. Buras, M. Misiak and J. Urban, Nucl. Phys. **B586** (2000) 397 [arXiv:hep-ph/0005183].
- [20] C. Bobeth *et al.*, JHEP **0404** (2004) 071 [arXiv:hep-ph/0312090].
- [21] K. Bieri, C. Greub and M. Steinhauser, Phys. Rev. **D67** (2003) 114019 [arXiv:hep-ph/0302051]; M. Misiak and M. Steinhauser, Nucl. Phys. **B683** (2004) 277 [arXiv:hep-ph/0401041].
- [22] N. Cabibbo, Phys. Rev. Lett. **10** (1963) 531; M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49** (1973) 652.
- [23] M. Beneke, T. Feldmann and D. Seidel, Nucl. Phys. **B612** (2001) 25 [arXiv:hep-ph/0106067]; H. M. Asatrian *et al.*, Phys. Rev. **D69** (2004) 074007 [arXiv:hep-ph/0312063].
- [24] A. J. Buras, P. Gambino and U. Haisch, Nucl. Phys. **B570** (2000) 117 [arXiv:hep-ph/9911250].
- [25] J. Collins, Renormalization, Cambridge University Press, New York 1984 and references therein.
- [26] H. Simma, Z. Phys. **C61** (1994) 67 [arXiv:hep-ph/9307274].
- [27] M. J. Dugan and B. Grinstein, Phys. Lett. **B256** (1991) 239; S. Herrlich and U. Nierste, Nucl. Phys. **B455** (1995) 39 [arXiv:hep-ph/9412375].
- [28] C. Becchi, A. Rouet and R. Stora, Phys. Lett. **B52** (1974) 344 and Commun. Math. Phys. **42** (1975) 127; I. V. Tyutin, LEBEDEV-75-39.
- [29] K. G. Chetyrkin, M. Misiak and M. Münz, Phys. Lett. **B400** (1997) 206, Erratum-ibid. **B425** (1998) 414 [arXiv:hep-ph/9612313].
- [30] H. D. Politzer, Nucl. Phys. **B172** (1980) 349.
- [31] T. Inami and C. S. Lim, Prog. Theor. Phys. **65** (1981) 297, Erratum-ibid. **65** (1981) 1772.

- [32] K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. **B192** (1981) 159; F. V. Tkachov, Phys. Lett. **B100** (1981) 65; D. J. Broadhurst, Z. Phys. **C54** (1992) 599; L. V. Avdeev *et al.*, Phys. Lett. **B336** (1994) 560, Erratum-ibid. **B349** (1995) 597 [arXiv:hep-ph/9406363]; J. Fleischer and O. V. Tarasov, Nucl. Phys. Proc. Suppl. **37B** (1994) 115 [arXiv:hep-ph/9407235]; L. V. Avdeev, Comput. Phys. Commun. **98** (1996) 15 [arXiv:hep-ph/9512442]; K. G. Chetyrkin, J. H. Kühn and M. Steinhauser, Phys. Lett. **B351** (1995) 331 [arXiv:hep-ph/9502291]; D. J. Broadhurst, Eur. Phys. J. **C8** (1999) 311 [arXiv:hep-th/9803091].
- [33] M. S. Chanowitz, M. Furman and I. Hinchliffe, Nucl. Phys. **B159** (1979) 225.
- [34] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. **B44** (1972) 189; P. Breitenlohner and D. Maison, Commun. Math. Phys. **52** (1977) 11, Commun. Math. Phys. **52** (1977) 39 and Commun. Math. Phys. **52** (1977) 55.
- [35] A. J. Buras, M. Jamin and M. E. Lautenbacher, Nucl. Phys. **B408** (1993) 209 [arXiv:hep-ph/9303284].
- [36] T. Muta, Foundations of Quantum Chromodynamics, Second Edition, World Scientific, Singapore 1998 and references therein.
- [37] M. Misiak, Nucl. Phys. **B393** (1993) 23, Erratum-ibid. **B439** (1995) 461.
- [38] M. Steinhauser, Comput. Phys. Commun. **134** (2001) 335 [arXiv:hep-ph/0009029].
- [39] A. J. Buras, Rev. Mod. Phys. **52** (1980) 199.